Drinfeld modular forms of higher rank from a lattice-oriented point of view

by

Liam Baker

Dissertation presented for the degree of Doctor of Philosophy in Mathematics in the Faculty of Science at Stellenbosch University

Supervisor: Dr Dirk Basson, Stellenbosch University
Co-supervisor: Prof Florian Breuer, University of Newcastle

March 2020

Declaration

By submitting this dissertation electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: March 2020

Abstract

Drinfeld modular forms of higher rank from a lattice-oriented point of view

L. Baker

Mathematics Division,
Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, 7602 Matieland, South Africa

Dissertation: PhD (Math)

March 2020

A space $\mathcal{L}_N^r$ of Drinfeld modules of rank $r \geq 1$ with level structure, or equivalently lattices of rank r with level structure, is introduced, and its irreducible components and group actions on it are investigated. A metric is defined on this space, its completion $\overleftarrow{\mathcal{L}_N^r}$ is established and the aforementioned group actions are extended to the completion. A decomposition of the completion into multiple smaller spaces $\mathcal{L}_N^s$ is proven. Drinfeld modular forms are defined as homogeneous holomorphic functions on $\mathcal{L}_N^r$ which are continuous on the completion $\overleftarrow{\mathcal{L}_N^r}$, and the group actions above are extended to actions on the spaces of modular forms. Finally, the modular forms defined here are compared with those of Basson, Breuer, and Pink, and it is shown that the cusp forms (those which are zero on the boundary) coincide.

Uittreksel

Drinfeld modular forms of higher rank from a lattice-oriented point of view

("Drinfeld modular forms of higher rank from a lattice-oriented point of view")

L. Baker

Wiskunde Afdeling,
Departement van Wiskundige Wetenskappe,
Universiteit van Stellenbosch,
Privaatsak X1, 7602 Matieland, Suid Afrika

Proefskrif: PhD (Wisk)

Maart 2020

'n Ruimte $\mathcal{L}^r_N$ van Drinfeld-modules van rang $r \geq 1$ met vlakstruktuur, of anders gestel roosters van rang r met vlakstruktuur, word bekendgestel, en die irreduseerbare komponente daarvan en groepsaksies daarop word ondersoek. 'n Metriek word op hierdie ruimte gedefinieer, die voltooiing daarvan word vasgestel as $\overleftarrow{\mathcal{L}^r_N}$ en die bogenoemde groepsaksies word uitgebrei tot die voltooiing. 'n Ontbinding van die voltooiing in verskeie kleiner ruimtes $\mathcal{L}^s_N$ is bewys. Drinfeld-modulêre vorms word gedefinieer as homogene holomorfiese funksies op $\mathcal{L}^r_N$ wat kontinu is op die voltooiing $\overleftarrow{\mathcal{L}^r_N}$, en die bogenoemde groepsaksies word uitgebrei tot op aksies op die ruimtes van modulêre vorms. Laastens word die modulêre vorms wat hier gedefinieer word, vergelyk met dié van Basson, Breuer en Pink, en dit word aangetoon dat die spitsvorms (dié wat nul op die grens is) saamval.

$\mathcal{S}\ \mathcal{D}\ \mathcal{G}$

We acknowledge the support of the Harry Crossley Foundation towards this research. Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to the Foundation.

The support of the DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to the CoE.

Contents

Introduction

Literature overview

In the beginning, Drinfeld defined what he called *elliptic modules* to prove a special case of the Langlands conjecture for GL_2 over function fields [Dri|en]. These modules are now called *Drinfeld modules*, and are similar to elliptic curves, although they have arbitrarily high rank $r \in \mathbb{N}$. In particular, Drinfeld constructed a moduli space of Drinfeld modules of rank r with level structure both as an algebraic variety and analytically as a double quotient of an $r-1$-dimensional space Ω^r, which is a rigid analytic space over a field $\mathbb{C}_\infty$ of positive characteristic.

There is, however, a natural definition of a *Drinfeld modular form* on Ω^r with values in $\mathbb{C}_\infty$ as given by Goss in [Gos80]; these can be defined algebraically à la Katz [Kat73] and analytically in analogy with classical modular forms, with Ω^r playing the role of the complex upper half plane.

In the case of rank 2, these modular forms are functions of one variable and are in closest analogy with classical modular forms, which only exist in rank 2. The bulk of the study of Drinfeld modular forms has thus focused on this case; for surveys of the developments in this area, see [Ge|DMC; Cor97a; Gek99].

In arbitrary rank, the next development was due to Kapranov [Kap|en] who constructed a compactification of the moduli variety of Drinfeld $\mathbb{F}_q[T]$-modules with level structure, which he used to prove finite dimensionality of the space of Drinfeld modular forms of any particular weight, as in [Gos92].

More recently, Basson, Breuer, and Pink wrote a series of papers [BBP1; BBP2; BBP3] establishing a theory of modular forms of arbitrary rank, building on the papers [Pin13; BR09; Bas17; BB17] and Basson's PhD thesis [Bas14], and followed by Pink's [Pin19]. In parallel, Gekeler has developed a theory of modular forms of arbitrary rank for the case of the simplest base ring $\mathbb{F}_q[T]$ in the series [GHR|1; GHR|2; GHR|3; GHR|4], where the connection with the Bruhat-Tits building $\mathcal{BT}$ is greatly used.

For a more detailed discussion of the history of Drinfeld modular forms of higher rank, see [BB17, Section 7].

Motivation

In the classical case of modular forms, the simplest case is that of Eisenstein series. For a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ and integer $k > 2$, the kth Eisenstein series is defined by

$$E^k(\Lambda) = \sum_{\lambda\in\Lambda}{}' \lambda^{-k} = \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\omega_1 + n\omega_2)^k}.$$

This series is most conceptually simply viewed as a homogeneous function of lattices, which is holomorphic in a suitable sense. However, it is most often normalised in the literature to $\omega_2 = 1$ using the homogeneity property, resulting in a function of one complex variable ω_1, which can then be studied using the well developed theory of complex functions. The homogeneity condition then transforms into a restriction of the behaviour of this function of one variable under the action of $\mathrm{GL}_2(\mathbb{Z})$.

In the case of the Drinfeld 'upper half plane', we can have lattices of arbitrarily high rank r:

$$\Lambda = A\omega_1 + A\omega_2 + \cdots + A\omega_r.$$

Eisenstein series and other modular forms can be defined similarly as functions of lattices, but if we normalise to make the last component $\omega_r = 1$ as before, we have a function of $r - 1$ variables, which is not as easily dealt with as the case $r = 2$. Other work that has been done on Drinfeld modules of higher rank (such as [BBP1; BBP2; BBP3] and [GHR|1; GHR|2; GHR|3; GHR|4]) has been done from this perspective of functions of $r - 1$ variables, whereas this thesis investigates modular forms as functions on the space of lattices.

As it is not as easily seen that the space of lattices can be given rigid analytic structure (so that one can reasonable speak of a holomorphic or analytic function on such a space), we first link this space to earlier work to carry over rigid analyticity proven there into our setting.*

This viewpoint yields some unexpected rewards:

- We define our modular forms of higher rank in a relatively 'low-tech' way, i.e. largely avoiding the use of modern algebraic geometry. Also, our final result Theorem 5.57 shows that our cusp forms are the same as those defined in other more 'high-tech' work. This may help those

*Unfortunately we were not able as of writing this thesis to prove that the space of lattices can be given rigid analytic structure in an intrinsic way, so we use a bijection with another space which is more easily given rigid analytic structure to accomplish this goal.

who wish to enter and make progress in this field without experience in algebraic geometry.

- We also find actions of the group $\mathcal{J}(A)$ of fractional ideals of A and the general linear group $\mathrm{GL}_r(A/N)$ for N an ideal of A on the spaces of modular forms of rank r. The latter action subsumes that of $\mathrm{GL}_r(A)$, which specialises to the subgroup of $\mathrm{GL}_r(A/N)$ with determinant in the base field $\mathbb{F}_q$, and that of $(A/N)^\times$, being the Galois group of the field $F(\zeta_N)$ of F with N-division points of the Carlitz module adjoined.

Outline of the thesis

In Chapter 1, we present an abridged introduction to Drinfeld modules (with and without level structure), presenting only the results necessary in later chapters. Similarly, in Chapter 2 we present an abridged introduction to lattices (with and without level structure), and also introduce the exponential function associated to a lattice, of which we prove some analytic properties.

In Chapter 3 we first present the well-known equivalence between lattices and Drinfeld modules, as well as their level structures. We then also present the realisation of the *moduli space* of lattices of rank r with level structure as a double quotient involving the Drinfeld period domain Ω^r and the ring of finite adeles $\mathbb{A}_F^{fin}$ originally proven by Drinfeld; we use this to derive a similar realisation for the space of lattices with level structure *itself*. We then investigate the decomposition of these spaces into irreducible components as detailed by Hubschmid, extending some results about the identification of these components, especially the 'identity component', and counting the number of components of each rank. In the final two sections of this chapter we investigate the actions of the general linear group $\mathrm{GL}_r(\hat{A})$ of profinite integers and the group $\left(\mathbb{A}_F^{fin}\right)^\times$ of invertible adeles on our spaces, which we specialise to their quotient actions of $\mathrm{GL}_r(A/N)$, for N an ideal of A, and $\mathcal{J}(A)$, the group of A-fractional ideals of F.

In Chapter 4, we first introduce metrics on the spaces of lattices with and without level structure and investigate their completions, as well as those of their irreducible components. Then we characterise these completions as unions of similar spaces of smaller rank, and also extend the group actions of $\mathrm{GL}_r(A/N)$ and $\mathcal{J}(A)$ defined earlier to these completions.

In Chapter 5, we first define weak and strong modular forms and cusp forms as homogeneous functions on the spaces of lattices with and without level structure, as well as the algebras consisting of these functions, and carry over the aforementioned group actions to actions on these spaces of modular forms. We then list some examples of modular forms and detail their behaviour under the above group actions. Finally we investigate the relation between our modular forms and those defined by Basson, Breuer, and Pink, showing our modular forms to be a large subset of theirs.

Finally, in Chapter 6 we present some closing remarks, including possible extensions to this work.

Notation

Throughout this thesis, we will make use of the following notation:

$\sum'$, $\min'$	A prime ($'$) used to denote a sum, product, minimum etc. over the *nonzero* elements of an index set.
$\#(S)$	The cardinality of a set S, also denoted $\#S$.
$\sqcup, \bigsqcup$	Disjoint union of sets.
$X - Y$	The complement of a set Y in a set X.
$\subseteq$, $\subset$	The subset and proper subset relations between sets, respectively.
$f^{-1}(y)$	The compositional inverse of a function.
$f(x)^{-1}$	The reciprocal of a function, $1/f(x)$.
$\mathrm{d}_M(z, S)$	For a metric space M with $z \in M$ and $S \subseteq M$, $$\mathrm{d}_M(z,S) \coloneqq \inf_{s\in S} \mathrm{d}_M(z,s).$$ The subscript M in d_M may be omitted if there is no ambiguity.
$\mathrm{d}_M(S_1, S_2)$	For subsets $S_1, S_2 \subseteq M$ of a metric space M, $$\mathrm{d}_M(S_1,S_2) \coloneqq \inf_{\substack{s_1\in S_1\\ s_2\in S_2}} \mathrm{d}_M(s_1,s_2).$$
∂S	The boundary of a subset S of a topological space.
$R^\times$	The multiplicative group of invertible elements of a ring R.
$(a)_R$	The principal ideal generated by an element a of a ring R, denoted by (a) if there is no ambiguity.

$\equiv_N$ The equivalence relation modulo N for an ideal N of a ring R. For an element a of R, the equivalence relation $\equiv_a$ is defined to be the same as the equivalence relation $\equiv_{(a)}$.

$\mathbb{N}$ The set of positive integers.

$\mathbb{N}_0$ The set of nonnegative integers.

$\mathbb{Z}$ The ring of integers.

$\mathbb{F}_g$ The finite field of cardinality g for g a prime power.

$\mathbb{P}^l(k)$ The l-dimensional projective space over a field k.

F A fixed global function field.

p The characteristic of F, a prime number.

q The cardinality of the field of constants of F, a power of p.

∞ A fixed place of F.

δ The degree of ∞.

A The ring of elements of F which are regular away from ∞.

$\mathcal{J}(A)$ The abelian group of fractional ideals of A.

$\mathcal{J}_{\geq 0}(A)$ The monoid of ideals of A. The notation (A) may be omitted from this and the previous item if not necessary.

$\mathrm{Cl}(F)$ The ideal class group of F.

$A_\mathfrak{p}$, $F_\mathfrak{p}$ The completions of A and F respectively at the prime $\mathfrak{p}$.

$\hat{A}$ The profinite completion of A, isomorphic to the product $\prod_{\mathfrak{p}\neq\infty} A_\mathfrak{p}$.

$\mathbb{A}_F^{fin}$ $\hat{A} \otimes_A F$, the ring of finite adeles of F, usually viewed as the restricted product $\widehat{\prod_{\mathfrak{p}\neq\infty}} F_\mathfrak{p}$ where each element $(x_\mathfrak{p})_\mathfrak{p}$ has $x_\mathfrak{p} \in A_\mathfrak{p}$ for almost all $\mathfrak{p}$.

$v_\mathfrak{p}$ The valuation associated to $\mathfrak{p}$ on A, F, $A_\mathfrak{p}$, $F_\mathfrak{p}$, $\hat{A}$ and $\mathbb{A}_F^{fin}$.

π A fixed uniformising parameter for F at ∞.

$\deg$ The degree function on F determined by $\deg \pi = -\delta$.

$|\cdot|$ The absolute value defined by $|x| := q^{\deg x}$ on F and the corres-

ponding unique extension to F_∞ and $\mathbb{C}_\infty$.
For nonzero $a \in A$, $|a| = \#(A/(a))$, and for an ideal N of A, $|N| = \#(A/N)$.
For a subset S of $\mathbb{C}_\infty$, $|S| \coloneqq \{|s| \mid s \in S\}$.

F_∞ The completion of F with respect to the metric induced by the absolute value $|\cdot|$, or equivalently the ∞-adic completion of F; isomorphic to $\mathbb{F}_{q^\delta}(\!(\pi)\!)$.

$\mathbb{C}_\infty$ The (metric) completion of an algebraic closure of F_∞.

ξ A fixed nonzero element of $\mathbb{C}_\infty$.

r A positive integer, called the *rank*.

V The space of all strongly discrete $\mathbb{F}_q$-sub-vector spaces of $\mathbb{C}_\infty$.

$\mathcal{L}^r, \mathcal{L}^{\le r}$ The *Drinfeld lattice domain of ranks r and $\le r$*; the spaces of lattices (A-submodules of finite rank) in $\mathbb{C}_\infty$ of rank r and $\le r$, respectively.

$\mathcal{L}^r_N$ The space of lattices in $\mathbb{C}_\infty$ of rank r with level N structure.

$\overleftarrow{\mathcal{L}^r_N}$ The space of lattices in $\mathbb{C}_\infty$ of rank at most r with r-inverse level N structure, defined in Definition 4.19.

Ω^r The *Drinfeld period domain of rank r*:

$$\left\{(\omega_1 : \omega_2 : \cdots : \omega_r) \in \mathbb{P}^{r-1}(\mathbb{C}_\infty) \,\middle|\, \omega_1, \ldots, \omega_r \text{ are } F_\infty\text{-lin.indep.}\right\},$$

considered as homogeneous row vectors or equivalently as equivalence classes of F-linear embeddings $\omega : F^r \hookrightarrow \mathbb{C}_\infty$ under scaling by $\mathbb{C}_\infty^\times$ where the images of the unit vectors in F^r are F_∞-linearly independent.

Ψ^r The *homogeneous* Drinfeld period domain of rank r:

$$\{(\psi_1, \psi_2, \ldots, \psi_r) \in \mathbb{C}_\infty^r \mid \psi_1, \ldots, \psi_r \text{ are } F_\infty\text{-lin.indep.}\},$$

considered as row vectors or equivalently as F-linear embeddings $\psi : F^r \hookrightarrow \mathbb{C}_\infty$ where the images of the unit vectors in F^r are F_∞-linearly independent.

1 Drinfeld modules with level

Drinfeld modules

1.1 Definition. We denote by $\mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty)$ the ring of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{C}_\infty$, with addition defined pointwise and multiplication defined as function composition. As a special element, we consider the *Frobenius endomorphism*:

$$\tau \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty), \quad X \mapsto X^q,$$

and we also consider the subring of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty)$ generated over $\mathbb{C}_\infty$ by τ, which we denote as $\mathbb{C}_\infty\{\tau\}$, and a superring of that, the ring of formal power series in τ with coefficients in $\mathbb{C}_\infty$, which we denote by $\mathbb{C}_\infty\{\{\tau\}\}$.

Each $f \in \mathbb{C}_\infty\{\{\tau\}\}$ can be uniquely written in the form $f = \sum_i l_i\tau^i$ (or equivalently $f(X) = \sum_i l_i X^{q^i}$); we then define $\mathrm{D}(f) = l_0$ (i.e. the 'constant term' of f) and $l(f) = l_{\deg f}$ for $f \in \mathbb{C}_\infty\{\tau\}$ (i.e. the 'leading coefficient' of f). This $D : \mathbb{C}_\infty\{\{\tau\}\} \to \mathbb{C}_\infty$ is a ring homomorphism.

1.2 Definition. A Drinfeld module of rank $r \geq 0$ is a ring homomorphism

$$\phi : A \to \mathbb{C}_\infty\{\tau\}, \quad a \mapsto \phi_a$$

such that for each $a \in A$, both

- $\deg_\tau \phi_a = r \cdot \deg a$ (here $\deg_\tau \phi_a$ denotes the degree of ϕ_a in τ), and
- $\mathrm{D}(\phi_a) = a$.

For two Drinfeld modules ϕ and φ of rank r, a morphism $u : \phi \to \varphi$ is an element of $\mathbb{C}_\infty\{\tau\}$ such that $u\phi_a = \varphi_a u$ for all $a \in A$; i.e. the diagram

1.3
$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{\phi_a} & \mathbb{C}_\infty \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ \mathbb{C}_\infty & \xrightarrow{\varphi_a} & \mathbb{C}_\infty \end{array}$$

commutes for each $a \in A$. A category of Drinfeld modules of rank r is thus formed in the natural way.

Note that since τ is $\mathbb{F}_q$-linear, so is each ϕ_a for a Drinfeld module ϕ. Hence each $f \in \mathbb{F}_q^\times$ is an automorphism of any Drinfeld module ϕ.

Also, the only Drinfeld module of rank $r = 0$ is the 'trivial' $\phi_a(X) = aX$ for all $a \in A$.

. . . with level structure

As before, each $f \in \mathbb{F}_q^\times$ is an automorphism of Drinfeld modules, considered as an element of $\mathbb{C}_\infty\{\tau\}$. However, some Drinfeld modules have nontrivial automorphisms (see the next chapter for an example). This is a problem since there is then no fine moduli space of Drinfeld modules, so we augment Drinfeld modules with *level structure* to remove these nontrivial automorphisms.

1.4 Definition. Let ϕ be a Drinfeld module. For an element $a \in A$, we define the set of a-division points $\phi[a] = \ker \phi_a$, and for an ideal N of A we define $\phi[N] = \bigcap_{a \in N} \phi[a]$.

For $a_1, a_2 \in A$, if $a_1 \mid a_2$ then $a_2 = b \cdot a_1$ for $b \in A$; thus $\phi_{a_2} = \phi_b \circ \phi_{a_1}$, and so $\phi[a_1] = \ker \phi_{a_1} \subseteq \ker \phi_{a_2} = \phi[a_2]$. So $\phi[(a)] = \phi[a]$, and the two parts of this definition agree.

1.5 Proposition. *$\phi[N]$ has the structure of an A/N-module, defined by*

$$a \cdot z = \phi_a(z) \quad \textit{for} \quad a \in A \quad \textit{and} \quad z \in \phi[N].$$

Proof. We must show that for any $a \in A$,

a) the map $z \mapsto \phi_a(z)$ sends elements of $\phi[N]$ into $\phi[N]$, and

b) the value $\phi_a(z)$ depends only on the class of a modulo N.

Here are the proofs of these statements:

a) For $b \in N$ and $z \in \phi[N]$, $\phi_b(\phi_a(z)) = \phi_{ab}(z) = \phi_a(\phi_b(z)) = 0$.

b) If $a_1 \equiv_N a_2$ and $z \in \phi[N]$, then $\phi_{a_1 - a_2}(z) = 0$, so $\phi_{a_1}(z) = \phi_{a_2}(z)$. $\square$

It is proven later that $\phi[N]$ is isomorphic to $(N^{-1}/A)^r$ as an A/N-module.

1.6 Definition. For an ideal N of A, a *level N structure* for a Drinfeld module ϕ of rank r is an A/N-module isomorphism

$$\beta : \left(N^{-1}/A\right)^r \xrightarrow{\sim} \phi[N].$$

1.7 Definition. We define the category of Drinfeld modules with level structure, where a morphism $u : (\phi, \beta) \to (\varphi, \beta')$ is an element of $\mathbb{C}_\infty\{\tau\}$ such that $u\phi_a = \varphi_a u$ for all $a \in A$ and $u \circ \beta = \beta'$. In other words, Diagram 1.3 commutes for each $a \in A$, and the following diagram also commutes:

1.8
$$\begin{tikzcd}
& \phi[N] \arrow[r, hook, "\subset"] & \mathbb{C}_\infty \arrow[dd, "u"] \\
(N^{-1}/A)^r \arrow[ur, hook, two heads, "\beta"] \arrow[dr, hook, two heads, "\beta'"'] & & \\
& \varphi[N] \arrow[r, hook, "\subset"'] & \mathbb{C}_\infty
\end{tikzcd}$$

With this definition of the category of Drinfeld modules with level structure, we finally have no more nontrivial automorphisms, as shown by the following proposition and corollary. However, in order to prove these results we will need some results from Chapter 3.

1.9 Proposition. *If $u : (\phi_1, \beta_1) \to (\phi_2, \beta_2) \in \mathbb{C}_\infty\{\tau\}$ is an isomorphism of Drinfeld modules with level N structure, then $u \in \mathbb{C}_\infty$ and $u\beta_1 = \beta_2$.*

Proof. Let (Λ_1, α_1) and (Λ_2, α_2) be the lattices with level structure corresponding to (ϕ_1, β_1) and (ϕ_2, β_2) respectively, with $u' \in \mathbb{C}_\infty$ the corresponding isomorphism of lattices with level N structure. Then by Proposition 2.11, $u'\Lambda_1 = \Lambda_2$ and $u'\alpha_1 = \alpha_2$. Translating back to Drinfeld modules, we see that by Proposition 3.1 and Proposition 2.15, $u'\phi_{1,a} = \phi_{2,a}u'$ for each $a \in A$ and $u'\beta_1 = \beta_2$; in other words, $u = u'$ is an element of $\mathbb{C}_\infty$ satisfying the desired conditions. □

1.10 Corollary. *If $u : (\phi, \beta) \to (\phi, \beta)$ is an automorphism of Drinfeld modules with level, then $u = 1$ (the identity morphism).*

Proof. See Corollary 2.13. □

2 Lattices with level

Lattices

2.1 Definition. An A-submodule $\Lambda \subset \mathbb{C}_\infty$ is called a lattice if and only if

1. Λ is finitely generated as an A-module, and
2. Λ is strongly discrete as a subset of $\mathbb{C}_\infty$ (i.e. any finite ball in $\mathbb{C}_\infty$ has finite intersection with Λ.)

The *rank* of Λ is its rank as a finitely generated torsion-free (or equivalently finitely generated projective) submodule of $\mathbb{C}_\infty$, the set of lattices of rank r is denoted $\mathcal{L}^r$, the set of lattices of rank $\leq r$ is denoted $\mathcal{L}^{\leq r}$, and the set of *all lattices* is denoted $\mathcal{L}$.

2.2 Definition. A prelattice is a strongly discrete $\mathbb{F}_q$-sub-vector space of $\mathbb{C}_\infty$.

Since A is an $\mathbb{F}_q$-vector space, any lattice is a prelattice.

2.3 Since A is a Dedekind domain, we have from [Go|Bas, Section 4.3] that if Λ is a lattice of rank $r \geq 1$, then there is an A-module isomorphism $\Lambda \simeq A^{r-1} \oplus I$ where I is a nonzero ideal of A. In other words, there are $\omega_1, \omega_2, \ldots, \omega_r \in \mathbb{C}_\infty$ such that $\Lambda = A\omega_1 + A\omega_2 + \cdots + A\omega_{r-1} + I\omega_r$ and the ω_i are F-linearly independent, since F is the fraction field of A. In fact, since Λ is strongly discrete, we must have that the ω_i are F_∞-linearly independent.

2.4 Definition. A morphism $c : \Lambda_1 \to \Lambda_2$ between two lattices *of the same rank* is an element $c \in \mathbb{C}_\infty$ such that $c\Lambda_1 \subseteq \Lambda_2$. The category of lattices is then defined in the natural way.

2.5 Note that for a morphism $c : \Lambda_1 \to \Lambda_2$ to have an inverse morphism c' in this category, we must have that $cc' = 1$ and $c\Lambda_1 \subseteq \Lambda_2$ and $c'\Lambda_2 \subseteq \Lambda_1$ (or equivalently $\Lambda_2 \subseteq c\Lambda_1$); thus $\Lambda_2 = c\Lambda_1$.

Every lattice has $\mathbb{F}_q^\times$ as a set of trivial automorphisms, but we also see that some special lattices have nontrivial automorphisms; for example, if

$f \in \mathbb{F}_{q^2} - \mathbb{F}_q \subset \mathbb{C}_\infty$, then $A + fA$ is a rank 2 lattice with f as an automorphism, since f satisfies a quadratic equation with coefficients in $\mathbb{F}_q \subset A$.

. . . with level structure

For this section, we let N be a fixed nonzero proper ideal of A.

2.6 Note that if $c : \Lambda_1 \to \Lambda_2$ is a nonzero morphism of lattices, then since $c\Lambda_1$ and Λ_2 have the same rank, $\Lambda_2/c\Lambda_1$ is a finite A-module. Also, for any nonzero $a \in A$ and lattice Λ, a is a morphism from Λ to itself.

Moreover, if Λ has rank r, then by [Go|Bas, p. 67] we have that

$$\Lambda/a\Lambda \simeq a^{-1}\Lambda/\Lambda \simeq \bigoplus_{i=1}^{r} A/(a)$$

is a finite $A/(a)$-module, and more generally if N is a nonzero ideal of A then

$$\Lambda/N\Lambda \simeq N^{-1}\Lambda/\Lambda \simeq \bigoplus_{i=1}^{r} A/N$$

is a free finite A/N-module. We can thus make the following

2.7 Definition. A *level N structure* for a lattice Λ of rank r is an A/N-module isomorphism

$$\alpha : \left(N^{-1}/A\right)^r \xrightarrow{\sim} N^{-1}\Lambda/\Lambda.$$

2.8 Proposition. *Every lattice of rank r has exactly $\#\mathrm{GL}_r(A/N)$ level N structures.*

Proof. Given Paragraph 2.6, we see that Λ possesses a level N structure. In fact, $\gamma \in \mathrm{GL}_r(A/N)$ acts from the right on the set of level N structures of Λ by $\alpha \overset{\gamma}{\mapsto} \alpha \circ \gamma$. Also, for any two level N structures α_1 and α_2, $\alpha_1^{-1} \circ \alpha_2$ is an A/N-module automorphism of N^{-1}/A, and hence an element of $\mathrm{GL}_r(A/N)$. Hence the group acts transitively. Finally, if $\alpha = \alpha \circ \gamma$, then since α is a bijection it follows that $\gamma = 1$. The result follows. $\square$

2.9 Definition. A morphism $c : (\Lambda_1, \alpha_1) \to (\Lambda_2, \alpha_2)$ between two pairs of a lattice of rank r with associated level N structure is an element $c \in \mathbb{C}_\infty$ such that both $c\Lambda_1 \subseteq \Lambda_2$ *and* the following diagram commutes:

2.10
$$\begin{tikzcd} N^{-1}\Lambda_1/\Lambda_1 \arrow[r, hook, two heads, "\times c"] & N^{-1}c\Lambda_1/c\Lambda_1 \arrow[d, hook, "\subseteq"] \\ (N^{-1}/A)^r \arrow[u, hook, two heads, "\alpha_1"] \arrow[r, hook, two heads, "\alpha_2"'] & N^{-1}\Lambda_2/\Lambda_2 \end{tikzcd}$$

Note that since the sets in the above diagram are finite, if we have such a morphism then the map on the right, induced from the inclusion of $c\Lambda_1$ in Λ_2, should also be a bijection. For this to be the case, we must have that if $\lambda \in c\Lambda_1$ is not in $cN\Lambda_1$, then it is not in $N\Lambda_2$; i.e. $N\Lambda_2 \cap c\Lambda_1 \subseteq cN\Lambda_1$. The reverse inclusion is apparent, so we must in fact have equality.

In contrast to the situation without level structure, here we have no nontrivial automorphisms:

2.11 Proposition. *If* $c : (\Lambda_1, \alpha_1) \xrightarrow{\sim} (\Lambda_2, \alpha_2)$ *is an isomorphism of lattices with level* N *structure, then* $c\Lambda_1 = \Lambda_2$ *and* $c\alpha_1 = \alpha_2$*, with* c *considered as an element of* $\mathbb{C}_\infty$*.*

Proof. As in Paragraph 2.5, we must have that $c\Lambda_1 = \Lambda_2$. Diagram 2.10 thus becomes the following:

2.12
$$\begin{tikzcd} N^{-1}\Lambda_1/\Lambda_1 \arrow[r, hook, two heads, "\times c"] & N^{-1}c\Lambda_1/c\Lambda_1 \arrow[d, hook, two heads, "="] \\ (N^{-1}/A)^r \arrow[u, hook, two heads, "\alpha_1"] \arrow[r, hook, two heads, "\alpha_2"'] & N^{-1}\Lambda_2/\Lambda_2 \end{tikzcd}$$

where the map on the right is the identity since $c\Lambda_1 = \Lambda_2$. Hence $c\alpha_1 = \alpha_2$. ☐

2.13 Corollary. *If* $c : (\Lambda, \alpha) \xrightarrow{\sim} (\Lambda, \alpha)$ *is an automorphism of lattices with level* N *structure, then* c *is the identity morphism.*

Proof. By Proposition 2.11, $c\Lambda = \Lambda$ and so considering the smallest element of Λ we see that $|c| = 1$, whence $|c - 1| \leq \max\{|c|, |1|\} = 1$. In this situation of an automorphism, Diagram 2.12 becomes the following:

$$\begin{tikzcd} N^{-1}\Lambda/\Lambda \arrow[r, hook, two heads, "\times c"] \arrow[dr, equal] & N^{-1}c\Lambda/c\Lambda \arrow[d, hook, two heads, "="] \\ (N^{-1}/A)^r \arrow[u, hook, two heads, "\alpha"] \arrow[r, hook, two heads, "\alpha"'] & N^{-1}\Lambda/\Lambda \end{tikzcd}$$

from which we can see that multiplication by c is the identity map on $N^{-1}\Lambda/\Lambda$. So multiplication by $c-1$ sends $N^{-1}\Lambda/\Lambda$ to zero; i.e. $(c-1)N^{-1}\Lambda \subseteq \Lambda$ or $(c-1)\Lambda \subseteq N\Lambda \subset \Lambda$, the latter inclusion being strict since N is a proper ideal.

Now let $z_0 \in \Lambda - (c-1)\Lambda$, and let $M = \{\lambda \in \Lambda \mid |\lambda| \leq |z_0|\} \ni z_0$, which is finite since Λ is strongly discrete. If $c - 1 \neq 0$, then

$$\begin{aligned}\#M &> \#\{\lambda \in (c-1)\Lambda \mid |\lambda| \leq |z_0|\} = \#\{\lambda \in \Lambda \mid |\lambda| \leq |z_0|/|c-1|\} \\ &\geq \#\{\lambda \in \Lambda \mid |\lambda| \leq |z_0|\} = \#M,\end{aligned}$$

a contradiction. Thus $c = 1$. □

By Proposition 2.11 we see that in the set of all lattices of rank r with level N structure, the isomorphism classes are each bijective to $\mathbb{C}_\infty^\times$ and are the orbits of the action of $\mathbb{C}_\infty^\times$ where $c \cdot (\Lambda, \alpha) = (c\Lambda, c\alpha)$.

(Pre-)Lattice-associated functions

There is also a special function in $\mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty)$ associated to each prelattice (i.e. strongly discrete $\mathbb{F}_q$-sub-vector space of $\mathbb{C}_\infty$; in particular, to each lattice), as follows:

2.14 Definition. For a prelattice Λ, the *exponential function* is

$$e_\Lambda(z) = z \cdot \prod_{\lambda \in \Lambda}{}' \left(1 - \frac{z}{\lambda}\right).$$

The above product converges as $|\lambda| \to \infty$ if Λ is infinite, since Λ is strongly discrete.

We collect the following properties of e_Λ, which we refrain from proving; for proofs, see [Go|Bas, Section 4.3; Ge|DMC, Section 2.2; BBP1, Chapter 2].

2.15 Proposition.

1. *If Λ is finite, then e_Λ is a polynomial.*
2. *e_Λ is an entire function on $\mathbb{C}_\infty$ with simple zeroes at and only at the points in Λ.*
3. $\frac{\mathrm{d}}{\mathrm{d}z} e_\Lambda(z) = 1.$
4. $\dfrac{1}{e_\Lambda(z)} = \displaystyle\sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}.$

5. *e_Λ is $\mathbb{F}_q$-linear.*
6. *For $c \in \mathbb{C}_\infty$, $e_{c\Lambda}(cz) = c \cdot e_\Lambda(z)$.*
7. *e_Λ has a power series of the following form, convergent for all $z \in \mathbb{C}_\infty$:*

$$e_\Lambda(z) = \sum_{i \in \mathbb{N}_0} e_{\Lambda,i} z^{q^i}.$$

8. *If $\Lambda_1 \subseteq \Lambda_2$, then $e_{\Lambda_1}(\Lambda_2)$ is also a prelattice, and*

$$e_{\Lambda_2}(z) = e_{e_{\Lambda_1}(\Lambda_2)}(e_{\Lambda_1}(z)).$$

The following property of e_Λ concerns its behaviour 'at infinity':

2.16 Proposition. *For variable $z \in \mathbb{C}_\infty$ and a prelattice Λ,*

$$\mathrm{d}(z, \Lambda) \to \infty \iff |e_\Lambda(z)| \to \infty.$$

Proof. For the forward direction, let $\mathrm{d}(z, \Lambda) = M$; then $|z - \lambda| \geq M$ for all $\lambda \in \Lambda$, and there is a $\lambda_0 \in \Lambda$ such that $|z - \lambda_0| = M$ since Λ is strongly discrete. Then

$$\begin{aligned} |e_\Lambda(z)| = |e_\Lambda(z - \lambda_0)| &= |z - \lambda_0| \cdot \prod_{\lambda \in \Lambda}' \left| 1 - \frac{z - \lambda_0}{\lambda} \right| \\ &= |z - \lambda_0| \cdot \prod_{\substack{\lambda \in \Lambda \\ |z - \lambda_0| \geq |\lambda|}}' \frac{|z - \lambda_0 - \lambda|}{|\lambda|} \\ &\geq M \cdot \prod_{\substack{\lambda \in \Lambda \\ |\lambda| \leq M}}' \frac{M}{|\lambda|} \geq M. \end{aligned}$$

Thus $|e_\Lambda(z)| \to \infty$ as $\mathrm{d}(z, \Lambda) \to \infty$, proving our forward claim.

For the backwards direction we instead prove the contrapositive; that if $\mathrm{d}(z, \Lambda)$ is bounded, then so is $|e_\Lambda(z)|$. To this end, as before let $\mathrm{d}(z, \Lambda) = m$ and $\lambda_0 \in \Lambda$ be such that $|z - \lambda_0| = m$. Then $e_\Lambda(z) = e_\Lambda(z - \lambda_0)$; but since e_Λ is entire, it is bounded on bounded sets, and so $e_\Lambda(z - \lambda_0)$ is bounded which proves the claim. $\square$

2.17 Corollary. *Let $\Lambda_1 \subseteq \Lambda_2$ be prelattices. Then for a variable $z \in \mathbb{C}_\infty$,*

$$\mathrm{d}(z, \Lambda_2) \to \infty \iff \mathrm{d}(e_{\Lambda_1}(z), e_{\Lambda_1}(\Lambda_2)) \to \infty.$$

Proof. By Proposition 2.15, $e_{\Lambda_2}(z) = e_{e_{\Lambda_1}(\Lambda_2)}(e_{\Lambda_1}(z))$. So by Proposition 2.16,

$$\begin{aligned} \mathrm{d}(z, \Lambda_2) \to \infty &\iff |e_{\Lambda_2}(z)| \to \infty \iff \left|e_{e_{\Lambda_1}(\Lambda_2)}(e_{\Lambda_1}(z))\right| \to \infty \\ &\iff \mathrm{d}(e_{\Lambda_1}(z), e_{\Lambda_1}(\Lambda_2)) \to \infty. \end{aligned}$$ □

The following property instead concerns the behaviour of e_Λ close to zero:

2.18 Proposition. *Let Λ be a prelattice with $\min'_{\lambda\in\Lambda}|\lambda| = R$. Then for $|z| < R$, $|e_\Lambda(z) - z| \leq |z|^q R^{1-q}$.*

Proof. We will use the following partial fraction decomposition, which can be proven by investigating the behaviour at each pole $z = f\lambda$:

$$\sum_{f\in\mathbb{F}_q^\times} \frac{1}{z - f\lambda} = \frac{(q-1)z^{q-2}}{z^{q-1} - \lambda^{q-1}}.$$

Recall that $1/e_\Lambda(z) = \sum_{\lambda\in\Lambda} 1/(z-\lambda)$. Since Λ is an $\mathbb{F}_q$-vector space, we further have that

$$\begin{aligned} \frac{-1}{e_\Lambda(z)} &= \frac{q-1}{e_\Lambda(z)} \\ &= \sum_{f\in\mathbb{F}_q^\times}\sum_{\lambda\in\Lambda} \frac{1}{z - f\lambda} = \sum_{\lambda\in\Lambda}\sum_{f\in\mathbb{F}_q^\times} \frac{1}{z - f\lambda} \\ &= \frac{q-1}{z} + {\sum_{\lambda\in\Lambda}}' \frac{(q-1)z^{q-2}}{z^{q-1} - \lambda^{q-1}} \\ \implies \left|\frac{1}{e_\Lambda(z)} - \frac{1}{z}\right| &\leq {\max_{\lambda\in\Lambda}}' \frac{|z^{q-2}|}{|z^{q-1} - \lambda^{q-1}|} \leq {\max_{\lambda\in\Lambda}}' \frac{|z|^{q-2}}{|\lambda|^{q-1}} = \frac{|z|^{q-2}}{R^{q-1}}. \end{aligned}$$

Since $|1/z| > |z|^{q-2}/R^{q-1}$, we have $|1/e_\Lambda(z)| = |1/z|$ and so $|e_\Lambda(z)| = |z|$. The previous calculation then concludes as desired:

$$|e_\Lambda(z) - z| = \left|e^\Lambda(z)\right||z|\left|\frac{1}{e_\Lambda(z)} - \frac{1}{z}\right| \leq |e_\Lambda(z)||z|^{q-1}R^{1-q} = |z|^q R^{1-q}$$ □

The following properties are valid for Λ a lattice (i.e. in addition to being a prelattice, also being an A-module of finite rank):

2.19 Proposition.

1. *For $a \in A$,* $\quad e_\Lambda(az) = a \cdot e_\Lambda(z) \cdot \displaystyle\prod'_{\lambda \in a^{-1}\Lambda/\Lambda} \left(1 - \frac{e_\Lambda(z)}{e_\Lambda(\lambda)}\right).$

2. *For an ideal N of A,* $\quad e_{N^{-1}\Lambda}(z) = e_\Lambda(z) \cdot \displaystyle\prod'_{\lambda \in N^{-1}\Lambda/\Lambda} \left(1 - \frac{e_\Lambda(z)}{e_\Lambda(\lambda)}\right).$

Proof. We prove the first assertion; the second can be proven similarly. For $z \notin a^{-1}\Lambda$, the ratio between the left-hand and right-hand sides is

$$
\begin{aligned}
\frac{\mathrm{RHS}}{\mathrm{LHS}} &= \frac{a \cdot e_\Lambda(z) \cdot \prod'_{\lambda \in a^{-1}\Lambda/\Lambda}\left(1 - \frac{e_\Lambda(z)}{e_\Lambda(\lambda)}\right)}{e_\Lambda(az)} \\
&= \frac{a \cdot e_\Lambda(z) \cdot \prod'_{\lambda_a \in a^{-1}\Lambda/\Lambda} \frac{e_\Lambda(\lambda_a - z)}{e_\Lambda(\lambda_a)}}{e_\Lambda(az)} \\
&= \frac{a \cdot z \cdot \prod'_{\lambda\in\Lambda}(1 - z/\lambda) \cdot \prod'_{\lambda_a \in a^{-1}\Lambda/\Lambda} \frac{\lambda_a - z}{\lambda_a} \prod'_{\lambda\in\Lambda} \frac{1-(\lambda_a - z)/\lambda}{1 - \lambda_a/\lambda}}{az \cdot \prod'_{\lambda\in\Lambda}(1 - az/\lambda)} {}^{*} \\
&= a \cdot \prod_{\lambda\in\Lambda} \frac{\lambda - z}{\lambda - az} \cdot \prod'_{\lambda_a \in a^{-1}\Lambda/\Lambda} \prod_{\lambda\in\Lambda} \frac{\lambda_a - \lambda - z}{\lambda_a - \lambda} \\
&= a \cdot \prod_{\lambda\in\Lambda} \frac{\lambda - z}{\lambda - az} \cdot \prod'_{\lambda_a \in a^{-1}\Lambda} \frac{\lambda_a - z}{\lambda_a} \Bigg/ \prod'_{\lambda\in\Lambda} \frac{\lambda - z}{\lambda} \\
&= a \cdot \prod_{\lambda\in\Lambda} \frac{\lambda - z}{\lambda - az} \cdot \prod'_{\lambda\in\Lambda} \frac{\lambda}{\lambda - z} \cdot \prod'_{\lambda\in\Lambda} \frac{\lambda/a - z}{\lambda/a} \\
&= a \cdot \prod_{\lambda\in\Lambda} \frac{\lambda - z}{\lambda - az} \cdot \prod'_{\lambda\in\Lambda} \frac{\lambda - az}{\lambda - z} \\
&= 1.
\end{aligned}
$$

The assertion is thus true for all $z \in \mathbb{C}_\infty$ by continuity. $\square$

2.20 Since e_Λ, being entire, is surjective, and $e_\Lambda(a) = e_\Lambda(b) \iff e_\Lambda(a-b) = 0 \iff a - b \in \Lambda$, we see that e_Λ is a bijection between $\mathbb{C}_\infty/\Lambda$ and $\mathbb{C}_\infty$. We will thus use the same notation $e_\Lambda(\lambda)$ both for $\lambda \in \mathbb{C}_\infty$ and for $\lambda \in \mathbb{C}_\infty/\Lambda$.

3 Lattices and Drinfeld modules

Equivalence between lattices and Drinfeld modules

In the previous two chapters, we introduced Drinfeld modules and lattices. As it turns out, these two concepts are intimately connected. Firstly, from each lattice we can construct an associated Drinfeld module:

3.1 Proposition. *If Λ is a lattice of rank r, then ϕ^Λ defined as follows is a Drinfeld module:*

$$\phi_a^\Lambda(X) = a \cdot X \prod_{\lambda \in a^{-1}\Lambda/\Lambda}{}' \left(1 - \frac{X}{e_\Lambda(\lambda)}\right) = a \cdot X \prod_{y \in e_\Lambda(a^{-1}\Lambda)}{}' \left(1 - \frac{X}{y}\right).$$

Proof. By Proposition 2.19, putting $X = e_\Lambda(z)$ we see that

$$\phi_a^\Lambda(e_\Lambda(z)) = a \cdot e_\Lambda(z) \cdot \prod_{\lambda \in a^{-1}\Lambda/\Lambda}{}' \left(1 - \frac{e_\Lambda(z)}{e_\Lambda(\lambda)}\right) = e_\Lambda(a \cdot z).$$

Thus for $a, b \in A$,

$$\phi_a^\Lambda(e_\Lambda(z)) + \phi_b^\Lambda(e_\Lambda(z)) = e_\Lambda(a \cdot z) + e_\Lambda(b \cdot z) = e_\Lambda((a+b) \cdot z) = \phi_{a+b}^\Lambda(e_\Lambda(z));$$

hence $\phi_a^\Lambda + \phi_b^\Lambda = \phi_{a+b}^\Lambda$ since e_Λ is surjective. Similarly, $\phi_a^\Lambda \circ \phi_b^\Lambda = \phi_{ab}^\Lambda$.

It is apparent that ϕ^Λ satisfies $\mathrm{D}(\phi_a^\Lambda) = a$, and

$$\begin{aligned} \deg_\tau \phi_a^\Lambda &= \log_q \deg_X \phi_a^\Lambda(X) \\ &= \log_q \#\left(a^{-1}\Lambda/\Lambda\right) = \log_q(\#(A/(a)))^r \\ &= r \cdot \log_q |a| = r \cdot \deg a. \end{aligned}$$

□

A fundamental result is that, in fact, *every* Drinfeld module arises from a lattice in this way. We state the following result without proof; for proof, see [Go|Bas, Theorem 4.6.9]:

3.2 Theorem. *Let ϕ be a Drinfeld module of rank r over $\mathbb{C}_\infty$. Then there is a lattice $\Lambda = \Lambda_\phi$ of rank r such that $\phi = \phi^\Lambda$. Moreover, the association $\phi \mapsto \Lambda_\phi$ gives rise to an equivalence of categories between Drinfeld modules of rank r and lattices of rank r.*

There is also an equivalence in the definitions of level structure for Drinfeld modules and lattices, as shown in the following propositions:

3.3 Proposition. *If a Drinfeld module ϕ corresponds to a lattice Λ, then*

$$\phi[a] = \big\{e_\Lambda(\lambda) \bigm| \lambda \in a^{-1}\Lambda/\Lambda\big\} = e_\Lambda(a^{-1}\Lambda) \quad \textit{and}$$
$$\phi[N] = \big\{e_\Lambda(\lambda) \bigm| \lambda \in N^{-1}\Lambda/\Lambda\big\} = e_\Lambda(N^{-1}\Lambda).$$

Proof. Follows from Proposition 3.1. □

In Proposition 3.1, we see the Drinfeld module polynomial $\phi_a^\Lambda(X)$ associated to a lattice Λ factorised as a product over $a^{-1}\Lambda/\Lambda$. In the same way, we can define Drinfeld module-associated polynomials for each ideal $N \subseteq A$:

3.4 Definition. If Λ is a lattice of rank r and N an ideal of A, then we define the polynomial

$$\phi_N^\Lambda(X) = X \prod_{\lambda \in N^{-1}\Lambda/\Lambda}{}' \left(1 - \frac{X}{e_\Lambda(\lambda)}\right) = X \prod_{y \in e_\Lambda(N^{-1}\Lambda)}{}' \left(1 - \frac{X}{y}\right).$$

3.5 Proposition. *For $z \in \mathbb{C}_\infty$, $\phi_N^\Lambda(e_\Lambda(z)) = e_{N^{-1}\Lambda}(z)$.*

Proof. Apply Proposition 2.19. □

3.6 Note that by Proposition 3.1 and Definition 3.4, the series of polynomials $\phi_N^\Lambda(X)$ and $\phi_a^\Lambda(X)$ are related by $\phi_a^\Lambda(X) = a \cdot \phi_{(a)}^\Lambda(X)$.

Note that above equivalence between lattices and Drinfeld modules also extends to an equivalence between lattices with level structure and Drinfeld modules with level structure, as follows:

3.7 Proposition. *If (Λ, α) is a lattice of rank r with level N structure, then $(\phi^\Lambda, e_\Lambda \circ \alpha)$ is a Drinfeld module of rank r with level N structure.*

Proof. For (Λ, α) a lattice of rank r with level N structure, by Proposition 3.3 we have that $e_\Lambda \circ \alpha$ is a bijection from $(N^{-1}/A)^r$ to $\phi[N]$. Moreover, since for every $\lambda \in N^{-1}\Lambda/\Lambda$ and $a \in A$ we have that $\phi_a(e_\Lambda(\lambda)) = e_\Lambda(a\lambda)$, the A-module structures on each side agree, and hence the A/N-module structures do too. □

Thus $\mathrm{GL}_r(A/N)$ acts from the right on the set of level N structures of a given Drinfeld module in a similar way as in Proposition 2.8.

The Drinfeld moduli space

3.8 Let

$$K(N) = \ker\big(\mathrm{GL}_r(\hat{A}) \twoheadrightarrow \mathrm{GL}_r(A/N)\big)$$

denote the principal congruence subgroup of level N, where

$$\hat{A} = \varprojlim_{J \in \mathcal{J}_{\geq 0}} A/J$$

is the profinite completion of A and N is a proper ideal of A.

We consider the moduli space of isomorphism classes of Drinfeld modules with level N structure, or equivalently of isomorphism classes of lattices with level N structure. By [Dri|en, Section 6; Pin13, p. 5], there is an algebraic variety $M^r_{A,K(N)}$ defined over F which acts as the aforementioned moduli space and an isomorphism

3.9 $$\mathrm{GL}_r(F)\backslash\big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)\big) \xrightarrow{\sim} M^r_{A,K(N)}(\mathbb{C}_\infty)$$

which sends the equivalence class of $(\omega, g) \in \Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{fin})$ to the isomorphism class of Drinfeld modules associated to the lattice $\Lambda = \omega\big(F^r \cap g\hat{A}^r\big)$* and the level structure α which makes the following diagram commute:

$$\begin{tikzcd}
(N^{-1}/A)^r \arrow[rrr, hook, two heads, "\alpha"] \arrow[d, hook, two heads, "\subset"'] & & & N^{-1}\Lambda/\Lambda \\
N^{-1}\hat{A}^r/\hat{A}^r \arrow[r, hook, two heads, "g"'] & N^{-1}g\hat{A}^r/g\hat{A}^r & & N^{-1}\big(F^r \cap g\hat{A}^r\big)/\big(F^r \cap g\hat{A}^r\big) \arrow[ll, hook, two heads, "\subset"'] \arrow[u, hook, two heads, "\omega"']
\end{tikzcd}$$

Here, the left and right action of $f \in \mathrm{GL}_r(F)$ and $k \in K(N)$ respectively on $(\omega, g) \in \Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{fin})$ is as follows:

$$f(\omega, g)k = (\omega f^{-1}, fgk).$$

Also, $\hat{A}^r$ and F^r are considered as sets of column vectors.

*Here we consider $\omega \in \Omega^r$ as a map $\omega : \mathbb{P}^r(F) \to \mathbb{C}_\infty$ in the obvious way.

3.10 In [Dri|en], Drinfeld requires the level N to lie in two distinct maximal ideals of A for his more general setting. Since we consider here Drinfeld modules and lattices over $\mathbb{C}_\infty$, we only require N to lie in one maximal ideal, as in [Pin13, p. 3]. Hence we only require $N \neq A$.

3.11 Definition. A group Γ acts *discontinuously* on a separable rigid analytic space Y if there is an index set I, an action of Γ on I, and an admissible covering $(Y_i)_{i \in I}$ of Y such that the following conditions are satisfied:

1. $\gamma Y_i = Y_{\gamma i}$ for $i \in I$, $\gamma \in \Gamma$
2. $\Gamma_i \coloneqq \{\gamma \in \Gamma \mid \gamma i = i\}$ is finite for each $i \in I$.
3. If $\gamma \notin \Gamma_i$ then $Y_i \cap Y_{\gamma i} = \emptyset$. Moreover, if $i, j \in I$ then $Y_j \cap Y_{\gamma i} = \emptyset$ for all but finitely many $\gamma \in \Gamma$.
4. For each $i \in I$, the covering $(Y_{\gamma i})_{\gamma \in \Gamma}$ of $\bigcup_{\gamma \in \Gamma} Y_{\gamma i}$ is admissible.

3.12 Proposition. *If a group Γ acts discontinuously on a separable rigid analytic space Y, then $\Gamma \backslash Y$ can be made into a separable rigid analytic space in such a way that the projection $\pi_{\Gamma, Y} : Y \twoheadrightarrow \Gamma \backslash Y$ is a morphism of rigid analytic spaces.*

Proof. See [Dri|en, p. 582]. □

3.13 Drinfeld showed in [Dri|en], as did Schneider and Stuhler in [SS91, §1], that the space Ω^r can be endowed with the structure of a separable rigid analytic space. Moreover, Drinfeld showed that any subgroup of $\mathrm{GL}_r(F)$ commensurable with $\mathrm{GL}_r(A)$ acts discontinuously on Ω^r; thus by Proposition 3.12 the quotient $\mathrm{GL}_r(F) \backslash \Omega^r$ can be given a derived rigid analytic structure. Thus the above double quotient can be given a rigid analytic structure, with the quotient $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)$ given the discrete topology.

3.14 The space $\mathcal{L}_N^r$ of lattices with level N structure has an action of $\mathbb{C}_\infty^\times$, given by scaling the lattice and the level structure, which is free by Corollary 2.13; thus each fibre of the quotient $\mathcal{L}_N^r \twoheadrightarrow \mathcal{L}_N^r/\mathbb{C}_\infty^\times$ is isomorphic to $\mathbb{C}_\infty^\times$. Moreover, by Proposition 2.11 the space of isomorphism classes of lattices with level N structure is the quotient $\mathcal{L}_N^r/\mathbb{C}_\infty^\times$.

We can extend the isomorphism in Equation 3.9 to an isomorphism between the set $\mathcal{L}_N^r$ and a related double quotient; but we will first define a rigid analytic structure on the space Ψ^r which will take the place of Ω^r:

3.15 Definition. We let $\mathcal{H}^r$ be the set of all hyperplanes in $\mathbb{C}_\infty^r$ which can be defined with coefficients in F_∞. Then we define the space

$$\Psi^r = \mathbb{C}_\infty^r - \bigcup_{H \in \mathcal{H}^r} H$$

of all points in $\mathbb{C}_\infty^r$ which do not lie on any F_∞-rational hyperplane.

There is an obvious analogy between this space Ψ^r and the traditional Ω^r, the latter being formed by deleting all F_∞-rational hyperplanes from $\mathbb{P}_r(\mathbb{C}_\infty)$. In fact, we use this analogy to define the rigid analytical structure on Ψ^r:

3.16 Proposition. *There is a bijection*

$$\begin{aligned} \kappa : \Omega^r \times \mathbb{C}_\infty^\times &\hookrightarrow\!\!\!\!\!\rightarrow \Psi^r \\ ((\omega_1 : \omega_2 : \ldots : \omega_r), \psi_r) &\mapsto (\omega_1, \omega_2, \ldots, \omega_r) \cdot \frac{\psi_r}{\omega_r} \\ (\psi_1, \psi_2, \ldots, \psi_r) &\leftarrow\!\shortmid ((\psi_1 : \psi_2 : \ldots : \psi_r), \psi_r) \end{aligned}$$

Proof. That there are no F_∞-rational relations on either side of the above map is easy to see; in particular, ψ_r and ω_r above are nonzero. By composing the above given map and its supposed inverse (which are well defined), we see that they are in fact inverses. □

This bijection is equivalent to normalising Ω^r so that the last component ω_r is equal to 1.

3.17 We thus define the rigid analytic structure on Ψ^r as the structure of the product $\Omega^r \times \mathbb{C}_\infty^\times$, each of these being rigid analytic spaces.

There is a left and right action of $f \in \mathrm{GL}_r(F)$ and $k \in K(N)$ respectively on $(\psi, g) \in \Psi^r \times \mathrm{GL}_r(\mathbb{A}_F^{fin})$, which extends that given in Paragraph 3.8, as follows:

$$f(\psi, g)k \coloneqq (\psi f^{-1}, fgk).$$

3.18 Proposition. *The above action of $f \in \mathrm{GL}_r(F)$ and $k \in K(N)$ respectively on $(\psi, g) \in \Psi^r \times \mathrm{GL}_r(\mathbb{A}_F^{fin})$ translates to $\Omega^r \times \mathbb{C}_\infty^\times \simeq \Psi^r$ via Proposition 3.16 as follows:*

$$f\big((\omega, \psi_r), g\big)k = \left(\left(\omega f^{-1}, \frac{(\omega f^{-1})_r}{\omega_r}\psi_r\right), fgk\right).$$

Here $(\omega f^{-1})_r$ denotes the last entry of ωf^{-1}, and the fraction $\frac{(\omega f^{-1})_r}{\omega_r}$ is independent of the representative for ω chosen in $\mathbb{C}_\infty^r$.

Proof. Let $\psi = \kappa(\omega, \psi_r)$ be a representative for ω in $\mathbb{C}_\infty^r$. Then

$$\begin{aligned} f\big((\omega,\psi_r),g\big)k &\overset{\kappa}{=} f(\psi,g)k = (\psi f^{-1}, fgk) \\ &\overset{\kappa^{-1}}{=} \big(\kappa^{-1}(\psi f^{-1}), fgk\big) \\ &= \big((\psi f^{-1},(\psi f^{-1})_r), fgk\big) = \left(\left(\psi f^{-1}, \frac{(\psi f^{-1})_r}{\psi_r}\psi_r\right), fgk\right) \\ &= \left(\left(\psi f^{-1}, \frac{(\omega f^{-1})_r}{\omega_r}\psi_r\right), fgk\right) \end{aligned}$$ □

Similarly to Paragraph 3.8 we then have a double quotient bijection for $\mathcal{L}_N^r$:

3.19 Theorem. *There is a bijection*

$$\Theta : \mathrm{GL}_r(F)\backslash\big(\Psi^r \times \mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)\big) \xrightarrow{\sim} \mathcal{L}_N^r$$

which sends the equivalence class of a pair $(\psi, g) \in \Psi^r \times \mathrm{GL}_r(\mathbb{A}_F^{fin})$ to the lattice $\Lambda = \psi\big(F^r \cap g\hat{A}^r\big)$[†] *and the level structure α which makes the following diagram commute:*

3.20
$$\begin{tikzcd} (N^{-1}/A)^r \arrow[rr, hook, two heads, "\alpha"] \arrow[d, hook, two heads, "\subset"'] & & N^{-1}\Lambda/\Lambda \\ N^{-1}\hat{A}^r/\hat{A}^r \arrow[r, hook, two heads, "g"'] & N^{-1}g\hat{A}^r/g\hat{A}^r & N^{-1}\big(F^r \cap g\hat{A}^r\big)/\big(F^r \cap g\hat{A}^r\big) \arrow[l, hook, two heads, "\subset"] \arrow[u, hook, two heads, "\psi"'] \end{tikzcd}$$

Proof. For $f \in \mathrm{GL}_r(F)$ and $k \in K(N)$,

$$\big(\psi f^{-1}\big)\big(F^r \cap (fgk)\hat{A}^r\big) = \psi\big(f^{-1}F^r \cap f^{-1}fg\hat{A}^r\big) = \psi\big(F^r \cap g\hat{A}^r\big),$$

so acting by $\mathrm{GL}_r(F)$ and $K(N)$ leaves the lattice $\Lambda = \psi\big(F^r \cap g\hat{A}^r\big)$ unchanged. Following the above commutative diagram, we see that the actions of $\mathrm{GL}_r(F)$ and $K(N)$ also leave the level structure α unchanged, since k changes nothing modulo N and the addition of f^{-1} on the right-hand map and f on the bottom left map cancel. Hence the above map is well defined.

If we consider the actions of $\mathbb{C}_\infty^\times$ on Ψ^r and $\mathcal{L}_N^r$ by scaling, their quotients are Ω^r and $M_{A,K(N)}^r(\mathbb{C}_\infty)$ respectively, each fibre being isomorphic to $\mathbb{C}_\infty^\times$. Moreover, this scaling commutes with the group actions of $\mathrm{GL}_r(F)$ and $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$ in the above double quotient. Hence the bijection

$$\mathrm{GL}_r(F)\backslash\big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)\big) \xrightarrow{\sim} M_{A,K(N)}^r(\mathbb{C}_\infty)$$

[†] Here we consider $\psi \in \Psi^r \subset \mathbb{C}_\infty^r$ as a map $F^r \to \mathbb{C}_\infty$ in the obvious way.

extends to a bijection

$$\mathrm{GL}_r(F)\backslash\big(\Psi^r \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big) \xrightarrow{\sim} \mathcal{L}_N^r. \qquad \square$$

3.21 Similarly to Drinfeld in [Dri|en] and Schneider and Stuhler in [SS91], we can show that Ψ^r can be given rigid analytic structure, and by extension the same for $\mathrm{GL}_r(F)\backslash\big(\Psi^r \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big)$, and thus also for $\mathcal{L}_N^r$, making the bijection in Theorem 3.19 a rigid analytic isomorphism.

3.22 From the above, we see that the set $\mathcal{L}_N^r$ of all lattices of rank r with level N structure can be given rigid analytic structure. From this we can induce rigid analytic structure on the set $\mathcal{L}^r$ of lattices of rank r *without* level structure, as follows:

3.23 Proposition. *$\mathcal{L}^r$ can be given a rigid analytic structure induced from that of $\mathcal{L}_N^r$.*

Proof. Consider the left action of $\mathrm{GL}_r(A/N)$ (considered as automorphisms of $(N^{-1}/A)^r$) on $\mathcal{L}_N^r$ defined by $\gamma(\Lambda,\alpha) = (\Lambda, \alpha\circ\gamma^{-1})$ for $\gamma \in \mathrm{GL}_r(A/N)$. This action is free since α and γ are bijections, and is transitive on the second component of (Λ,α) while leaving the first unchanged; hence the quotient $\mathrm{GL}_r(A/N)\backslash\mathcal{L}_N^r$ is bijective with $\mathcal{L}^r$. Now, using the result of Proposition 3.12, since $\mathrm{GL}_r(A/N)$ is finite it acts discontinuously on $\mathcal{L}_N^r$ and so its quotient $\mathcal{L}^r$ has an induced rigid analytic structure. $\square$

Irreducible components of $M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathcal{L}_N^r$

The rigid analytic space $M^r_{A,K(N)}(\mathbb{C}_\infty)$ decomposes into irreducible components as in [Hub13, Proposition 2.1.3], given below. We call the corresponding partition of $\mathcal{L}_N^r$, induced from its quotient map onto $M^r_{A,K(N)}(\mathbb{C}_\infty)$, the irreducible components of $\mathcal{L}_N^r$.

3.24 Proposition. *Let H be a set of representatives in $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$ for the double quotient $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)$, and set $\Gamma_g = gK(N)g^{-1}\cap\mathrm{GL}_r(F)$ for $g \in H$. Then the map*

$$\begin{aligned}\bigsqcup_{g\in H}\Gamma_g\backslash\Omega^r &\longrightarrow \mathrm{GL}_r(F)\backslash\big(\Omega^r\times\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big)\\ [\omega]_g &\longmapsto [(\omega,g)]\end{aligned}$$

is a rigid analytic isomorphism which maps for each $g\in H$ the quotient space $\Gamma_g\backslash\Omega^r$ to an irreducible component of $M^r_{A,K(N)}(\mathbb{C}_\infty)$.

Each Γ_g is an arithmetic subgroup of $\mathrm{GL}_r(F)$.

Here is the corresponding result for $\mathcal{L}_N^r$:

3.25 Proposition. *For H and Γ_g as in Proposition 3.24, the map*

$$\begin{aligned} \bigsqcup_{g \in H} \Gamma_g \backslash \Psi^r &\longrightarrow \mathrm{GL}_r(F) \backslash \big(\Psi^r \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big) \\ [\psi]_g &\longmapsto [(\psi, g)] \end{aligned}$$

is a rigid analytic isomorphism which maps for each $g \in H$ the space $\Gamma_g \backslash \Psi^r$ to an irreducible component of $\mathcal{L}_N^r$. Here the action of $f \in \Gamma_g \subseteq \mathrm{GL}_r(F)$ on $\psi \in \Psi^r$ is as in Theorem 3.19, i.e. $f \cdot \psi = \psi f^{-1}$.

Proof. Consider the action of $\mathbb{C}_\infty^\times$ on Ψ^r, with quotient Ω^r, each fibre of which is isomorphic to $\mathbb{C}_\infty^\times$; since this action of $\mathbb{C}_\infty^\times$ commutes with the actions of $\mathrm{GL}_r(F)$ and $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$, the bijection in Proposition 3.24 extends to the bijection given above. □

We include the following result from [Hub13, Definition 3.4.1, Proposition 3.4.2]:

3.26 Proposition. *The map* det *from $M_{A,K(N)}^r(\mathbb{C}_\infty)$ given by*

$$\begin{aligned} \mathrm{GL}_r(F) \backslash \big(\Omega^r \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big) &\to F^\times \Big\backslash \big(\mathbb{A}_F^{fin}\big)^\times \Big/ \det K(N) \\ [(\omega, g)] &\mapsto [\det g] \end{aligned}$$

is surjective and the fibres are the irreducible components of $M_{A,K(N)}^r(\mathbb{C}_\infty)$.

3.27 Corollary. *The map* det *above induces a bijection*

$$\begin{aligned} \mathrm{GL}_r(F) \backslash \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N) &\hookrightarrow\!\!\!\!\!\rightarrow F^\times \Big\backslash \big(\mathbb{A}_F^{fin}\big)^\times \Big/ \det K(N) \\ [g] &\mapsto [\det g]. \end{aligned}$$

We know the number of irreducible components by [Hub13, Corollary 3.4.5]:

3.28 Proposition. *The irreducible components of $\mathcal{L}_N^r$ have number*

$$\#\Big(F^\times \Big\backslash \big(\mathbb{A}_F^{fin}\big)^\times \Big/ \det K(N) \Big) = \#\mathrm{Cl}(F) \cdot \#\big(\hat{A}^\times / (\mathbb{F}_q^\times \cdot \det K(N))\big).$$

We can actually take the above count further as shown in Proposition 3.30:

3.29 Lemma. $\det K(N) = \hat{A}^\times \cap (1 + N\hat{A}) = \prod_{\mathfrak{p}\nmid N} A_{\mathfrak{p}}^\times \cdot \prod_{\mathfrak{p}\mid N} 1 + (\mathfrak{p}A_{\mathfrak{p}})^{v_{\mathfrak{p}}(N)}$.

Proof. Each $x \in \det K(N)$ has $x \equiv_N 1$, and conversely if $x \in \hat{A}^\times \cap (1 + N\hat{A})$ then $x' \in \mathrm{GL}_r(\hat{A})$, which viewed as an $r \times r$ matrix has x in the first entry, 1 along the rest of the diagonal and 0 elsewhere, has $\det x' = x$ and is in $K(N)$. This proves the first equality.

For the second equality, note that

$$\begin{aligned} x = (x_{\mathfrak{p}})_{\mathfrak{p}} \equiv_N 1 &\iff x - 1 \in N\hat{A} \\ &\iff x - 1 \in \mathfrak{p}^{v_{\mathfrak{p}}(N)}\hat{A} && \text{for each } \mathfrak{p} \mid N \\ &\iff x_{\mathfrak{p}} - 1 \in (\mathfrak{p}A_{\mathfrak{p}})^{v_{\mathfrak{p}}(N)} && \text{for each } \mathfrak{p} \mid N \end{aligned}$$ □

3.30 Proposition. *The injective map*

$$i_N : \begin{aligned} (A/N)^\times / \mathbb{F}_q^\times &\hookrightarrow F^\times \Big\backslash \left(\mathbb{A}_F^{fin}\right)^\times \Big/ \det K(N) \\ [x, x \in A] &\mapsto [(x)_{\mathfrak{p}\mid N} \cup (1)_{\mathfrak{p}\nmid N}] \end{aligned}$$

and the surjective map

$$\pi_N : \begin{aligned} F^\times \Big\backslash \left(\mathbb{A}_F^{fin}\right)^\times \Big/ \det K(N) &\twoheadrightarrow \mathrm{Cl}(F) \\ [x] &\mapsto \left[\prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x)}\right]_{\mathrm{Cl}(F)} \end{aligned}$$

together form a short exact sequence of abelian groups.

Proof. First we show that i_N and π_N are well defined and are injective and surjective respectively. Note that $v_{\mathfrak{p}}(k) = 0$ for all prime $\mathfrak{p}$ and $k \in \det K(N)$.

i_N: Let $x, y \in A$ with $x + N, y + N \in (A/N)^\times$, noting that $v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}(y) = 0$ for all $\mathfrak{p} \mid N$. Then for $f \in \mathbb{F}_q^\times$,

$$\begin{aligned} & x \equiv fy \pmod{N} \\ \iff & x/fy \equiv 1 \pmod{N} \\ \iff & x/fy \equiv 1 \pmod{\mathfrak{p}^{v_{\mathfrak{p}}(N)}} \quad \text{for all } \mathfrak{p} \mid N \\ \iff & ((x)_{\mathfrak{p}\mid N} \cup (1)_{\mathfrak{p}\nmid N})/f((y)_{\mathfrak{p}\mid N} \cup (1)_{\mathfrak{p}\nmid N}) \in \det K(N); \end{aligned}$$

thus i_N is well-defined and injective.

π_N: Let $x = \left(\mathbb{A}_F^{fin}\right)^\times$, $f \in F^\times$, and $k \in \det K(N)$. Then $\prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}(f)} = (f)$ is principal, so that $\left[\prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}(x)}\right] = \left[\prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}(fxk)}\right]$. Also, if I is a fractional ideal then $v_\mathfrak{p}(I) \neq 0$ for only finitely many $\mathfrak{p}$; choosing uniformisers $u_\mathfrak{p} \in F_\mathfrak{p}$ for all such, we have that π_N maps $\left[\left(u_\mathfrak{p}^{v_\mathfrak{p}(I)}\right)_{v_\mathfrak{p}(I)\neq 0} \cup (1)_{v_\mathfrak{p}(I)=0}\right]$ to $[I]$.

Now to show that $\operatorname{Im} i_N = \ker \pi_N$, let $[x] \in \ker \pi_N$, where $x = (x_\mathfrak{p})_\mathfrak{p}$. Then $\prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}(x)} = (f)$ is principal, for some $f \in F^\times$. Thus $v_\mathfrak{p}(x/f) = 0$ for all $\mathfrak{p}$, so that $x/f \in \hat{A}^\times$ and so is invertible under the projection $\hat{A} \twoheadrightarrow A/N$. So there is a $t \in (A/N)^\times$ such that $x/f \equiv t \pmod N$, or equivalently $x/f \equiv (t)_{\mathfrak{p}|N} \cup (1)_{\mathfrak{p}\nmid N} \pmod N$, so we have that $k = (x/f)/((t)_{\mathfrak{p}|N} \cup (1)_{\mathfrak{p}\nmid N}) \in \det K(N)$. Thus $[x] = [f \backslash x/k] \in \operatorname{Im} i_N$. Conversely, consider $x = (t)_{\mathfrak{p}|N} \cup (1)_{\mathfrak{p}\nmid N} \in \hat{A}^\times$ for $t \in A$ which is invertible modulo N, so that $[x] \in \operatorname{Im} i_N$. Then $v_\mathfrak{p}(x) = 0$ for all $\mathfrak{p}$, so that $\pi_N(x) = 0$, i.e. $x \in \ker \pi_N$. □

So for each irreducible component C of $\mathcal{L}_N^r$, there is a corresponding class group element $\pi_N(C) \in \mathrm{Cl}(F)$, and those for which $\pi_N(C) = 1$ can be written as $C = i_N(x)$ for some $x \in (A/N)^\times/\mathbb{F}_q^\times$. Note here the abuse of notation: we will use π_N as a function from $F^\times \Big\backslash \left(\mathbb{A}_F^{fin}\right)^\times \Big/ \det K(N)$, from the double quotients $\mathrm{GL}_r(F)\backslash\left(\Omega^r \times \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right)/K(N)\right)$ and $\mathrm{GL}_r(F)\backslash\left(\Psi^r \times \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right)/K(N)\right)$, and from the set of irreducible components. Similarly, i_N could have as codomain any of the above spaces, with context dictating which is intended.

3.31 Definition. The *identity component* of $M^r_{A,K(N)}(\mathbb{C}_\infty)$ is the fibre of the identity element in the surjection of Proposition 3.26. We will use the notation 1_N^r for the corresponding identity component of $\mathcal{L}_N^r$.

So far we have looked at identifying the different components from the point of view of $\omega \in \Omega^r$ and $g \in \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right)$. The following series of results carries through this identification to the point of view of a lattice Λ with level structure α.

3.32 Proposition. *For fractional ideals $I_1, \ldots, I_r$ of F and $\psi = (\psi_1, \ldots, \psi_r) \in \Psi^r$, and the resulting lattice $\Lambda = I_1\psi_1 + \cdots + I_r\psi_r$ together with any associated level structure α, we have that $\pi_N(\Lambda, \alpha) = [I_1 I_2 \cdots I_r]_{\mathrm{Cl}(F)} \in \mathrm{Cl}(F)$.*

Proof. We will proceed by finding ψ' and g such that $\Theta([(\psi', g)]) = (\Lambda, \alpha)$, with Θ being the isomorphism

$$\Theta : \mathrm{GL}_r(F)\backslash\left(\Psi^r \times \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right)/K(N)\right) \xrightarrow{\sim} \mathcal{L}_N^r$$

from Theorem 3.19. We choose $\psi' = \psi$, and proceed to constructing g. Each I_i can be uniquely factorised as a product $\prod_{\mathfrak{p}} \mathfrak{p}^{a_{\mathfrak{p},i}}$ for prime ideals $\mathfrak{p}$ and integers $a_{\mathfrak{p},i}$ almost all zero, so choosing uniformisers $u_{\mathfrak{p}}$ for each localisation $A_{\mathfrak{p}}$ with $a_{\mathfrak{p},i}$ not all zero we have that $g_i := \left(u_{\mathfrak{p}}^{a_{\mathfrak{p},i}}\right)_{\mathfrak{p}}$ satisfies $g_i\hat{A} = I_i\hat{A}$. Let g' be the matrix with the g_i on the diagonal and zeroes elsewhere. We have that

$$g'\hat{A}^r = \left(g_1\hat{A}, \dots, g_r\hat{A}\right)^T = \left(I_1\hat{A}, \dots, I_r\hat{A}\right)^T,$$

so that

$$\psi\left(F^r \cap g'\hat{A}^r\right) = \psi(I_1, \dots, I_r) = I_1\psi_1 + \cdots + I_r\psi_r = \Lambda$$

as desired. Now this g' induces a level structure $\alpha' : \left(N^{-1}/A\right)^r \hookrightarrow\!\!\!\!\rightarrow N^{-1}\Lambda/\Lambda$ as in Theorem 3.19, which may not be the desired α. However, since $\mathrm{GL}_r\left(\hat{A}\right)$ surjects onto $\mathrm{GL}_r(A/N)$, there is a lift $\gamma \in \mathrm{GL}_r\left(\hat{A}\right)$ of $\alpha'^{-1} \circ \alpha \in \mathrm{GL}_r(A/N)$. Then defining $g = g' \circ \gamma$, we have that $g\hat{A}^r = g'\hat{A}^r$, so that $\Lambda = \psi\left(F^r \cap g\hat{A}^r\right)$, and by following Diagram 3.20 that g induces the level structure α.

Now $\det \gamma \in \hat{A}^\times$, so $\det g = \det g' \det \gamma \in g_1 \cdots g_r \hat{A}^\times$. Thus

$$\begin{aligned}
\pi_N(\Lambda, \alpha) &= \pi_N(\det g) \\
&= \left[\prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(g_1 \cdots g_r)}\right] = \left[\prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(g_1)} \cdots \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(g_r)}\right] \\
&= [I_1 \cdots I_r].
\end{aligned}$$

□

3.33 Corollary. *$\pi_N(\Lambda, \alpha) = [A]$ if and only if $\Lambda = A\psi_1 + \cdots A\psi_r = \psi A^r$ for some row vector $\psi = (\psi_1, \dots, \psi_r) \in \Psi^r$.*

Proof. For the forward direction, by Paragraph 2.3 there are an ideal I of A and $\psi = (\psi_1, \dots, \psi_r) \in \Psi^r$ such that $\Lambda = A\psi_1 + \cdots + A\psi_{r-1} + I\psi_r$. Now by Proposition 3.32 we have $[I] = [A]$ in $\mathrm{Cl}(F)$, i.e. $I = (n)$ is principal for some $n \in A$. Hence replacing ψ_r' by $n\psi_r$, we have that $\Lambda = A\psi_1 + \cdots + A\psi_r = \psi A^r$.

The converse is a direct application of Proposition 3.32. □

3.34 Note that the value of $\pi_N(\Lambda, \alpha)$ does not depend at all on the level structure α or even the ideal N; hence we may make use of the notation $\pi(\Lambda)$ instead, and in fact will also use $\pi(C)$ to denote $\pi(\Lambda)$ for a lattice in the irreducible component C.

3.35 Definition. For a lattice Λ with $\pi(\Lambda) = [A]$, to each choice of a generating vector $\psi \in \Psi^r$ satisfying $\Lambda = \psi A^r$ there is an associated canonical level N structure $\alpha_\psi : (N^{-1}/A)^r \hookrightarrow\!\!\!\!\rightarrow N^{-1}\Lambda/\Lambda$, given by $\alpha_\psi(l) = \psi l$[‡] for $l_i \in (N^{-1}/A)^r$.

The choice of a canonical level N structure α_ψ for a lattice $\Lambda = \psi A^r$ is equivalent to having $(\Lambda, \alpha_\psi) = \Theta([(\psi, \mathrm{Id})])$ where $\mathrm{Id} \in \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$ is the identity matrix.

3.36 Note that for any two choices $\psi_1, \psi_2 \in \Psi^r$ of generating vectors for a lattice Λ with $\pi(\Lambda) = [A]$, since $\psi_1 A^r = \psi_2 A^r$ we have that $\psi_2 = \psi_1\gamma$ for some $\gamma \in \mathrm{GL}_r(A)$, and hence $\alpha_{\psi_2}(l) = \psi_2 l = \psi_1\gamma l = \alpha_{\psi_1}(\gamma l)$ for all $l \in (N^{-1}/A)^r$, i.e. $\alpha_{\psi_2} = \alpha_{\psi_1} \circ \gamma$. Thus $\det \alpha_{\psi_1}^{-1} \circ \alpha_{\psi_2} \in A^\times = \mathbb{F}_q^\times$.

3.37 Proposition. *Let* $(\Lambda, \alpha) = \Theta([(\psi, g)])$. *Then if* $\pi(\Lambda) = [A]$, *we have that* $i_N([\det \alpha_\psi^{-1} \circ \alpha]) = [\det g]$.

Proof. We may choose different representatives ψ and g for (Λ, α), since $[\det g]$ is invariant under such a change and

$$[\det \alpha_{\psi_2}^{-1} \circ \alpha] = [\det \alpha_{\psi_2}^{-1} \circ \alpha_{\psi_1}^{-1}][\det \alpha_{\psi_1}^{-1} \circ \alpha] = [\det \alpha_{\psi_1}^{-1} \circ \alpha]$$

for any two generating vectors ψ_1, ψ_2 for Λ.

Now since $\pi(\Lambda) = [A]$, there is a generating vector $\psi \in \Psi^r$ for Λ. Choosing $g' = \mathrm{Id}$ to be the identity matrix, $\Theta([(\psi, g')]) = (\Lambda, \alpha_\psi)$. Letting $\gamma \in \mathrm{GL}_r\big(\hat{A}\big)$ be a lift of $\alpha_\psi^{-1} \circ \alpha \in \mathrm{GL}_r(A/N)$, we can define $g = g'\gamma = \gamma$ which by following Diagram 3.20 we see induces our level structure α. So $\alpha_\psi^{-1} \circ \alpha = \gamma \bmod N$, so that

$$[\det g] = [\det \gamma] = i_N([\det \alpha_\psi^{-1} \circ \alpha]). \qquad \square$$

3.38 Theorem. $(\Lambda, \alpha) \in 1_N^r$ *if and only if there is a generating vector* $\psi \in \Psi^r$ *for* Λ *such that* $\alpha = \alpha_\psi$.

Proof. For the forward direction, let (Λ, α) be in the identity component 1_N^r. Then by Corollary 3.33 there is a $\psi' = (\psi_1', \ldots, \psi_r') \in \Psi^r$ such that $\Lambda = \psi' A^r$.

Now by Proposition 3.37, since (Λ, α) is in 1_N^r we have that $\det \alpha_{\psi'}^{-1} \circ \alpha \in \mathbb{F}_q^\times$, and so there is a lift $\gamma \in \mathrm{GL}_r(A)$ for $\alpha_{\psi'}^{-1} \circ \alpha \in \mathrm{GL}_r(A/N)$. So define $\psi = \psi'\gamma$; then $\psi A^r = \psi'\gamma A^r = \psi' A^r = \Lambda$, and for $l \in (N^{-1}/A)^r$ we have that $\alpha(l) = \alpha_{\psi'}(\gamma l) = \psi'\gamma l = \psi l$.

[‡]Here ψ is considered as a row vector and l as a column vector.

For the reverse direction, note that $(\Lambda, \alpha) = \Theta([(\psi, \mathrm{Id})])$ where Id is the identity in $\mathrm{GL}_r(\mathbb{A}_F^{fin})$, and hence (Λ, α) lies in 1_N^r. $\square$

In other words, (Λ, α) is in the identity component if and only if $\pi(\Lambda) = [A]$ and a canonical level structure is used.

The action of $\mathrm{GL}_r(A/N)$ on $M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathcal{L}_N^r$

3.39 Definition. We define a left action of $\mathrm{GL}_r(\hat{A})$ on $M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathcal{L}_N^r$ by

$$\gamma[(\omega, g)] = [(\omega, g \circ \gamma^{-1})] \quad \text{and} \quad \gamma[(\psi, g)] = [(\psi, g \circ \gamma^{-1})] \quad \text{for } \gamma \in \mathrm{GL}_r(\hat{A}).$$

3.40 Proposition. *The above action is well defined.*

Proof. If $[(\psi_1, g_1)] = [(\psi_2, g_2)]$ in (the double quotient isomorphic to) $\mathcal{L}_N^r$ then there are $f \in \mathrm{GL}_r(F)$ and $k \in K(N)$ such that $\psi_2 = \psi_1 f^{-1}$ and $g_2 = f g_1 k$. Thus $g_2\gamma^{-1} = f g_1 k \gamma^{-1} = f g_1 \gamma^{-1}(\gamma k \gamma^{-1})$ with $\gamma k \gamma^{-1} \in K(N)$, so that $[(\psi_2, g_2\gamma^{-1})] = [(\psi_1, g_1\gamma^{-1})]$. The proof for $M^r_{A,K(N)}(\mathbb{C}_\infty)$ is similar. $\square$

3.41 Note that since the above action leaves the components of Ω^r and Ψ^r unchanged, and the topology on the quotient $\mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)$ is discrete, the above actions are rigid analytic automorphisms of the relevant spaces.

3.42 Proposition. *The kernel of the above action on $\mathcal{L}_N^r$ is the normal subgroup $K(N) \triangleleft \mathrm{GL}_r(\hat{A})$; it thus induces an action of $\mathrm{GL}_r(A/N) \simeq \mathrm{GL}_r(\hat{A})/K(N)$.*

Proof. Firstly, let $\Omega^r \times \mathbb{C}_\infty^\times \ni (\omega, \psi_r) = \kappa^{-1}(\psi)$ for $\psi \in \Psi^r$. Then for any $f \in \mathrm{GL}_r(F)$ and a representative $\varpi \in \Psi^r$ for $\omega \in \Omega^r$,

$$\begin{aligned}
&\psi = f\psi \\
\iff &(\omega, \psi_r) = f(\omega, \psi_r) = \left(\omega f^{-1}, \frac{(\omega f^{-1})_r}{\omega_r}\psi_r\right) \\
\iff &\omega = \omega f^{-1} \quad \text{and} \quad \psi_r = \frac{(\varpi f^{-1})_r}{\varpi_r}\psi_r \\
\iff &\omega = \omega f^{-1} \quad \text{and} \quad (\varpi f^{-1})_r = \varpi_r \\
\iff &\varpi = \varpi f^{-1} \\
\iff &(f-1)\varpi = 0 \\
\iff &f = 1
\end{aligned}$$

since the $\overline{\omega}_i$ are F_∞-linearly independent. Thus

$$\begin{aligned}
&\gamma \in \mathrm{GL}_r\big(\hat{A}\big) \text{ is in the kernel of the action}\\
\iff &(\forall[(\psi,g)] \in \mathcal{L}^r_N)\ [(\psi,g)] = \gamma[(\psi,g)] = [(\psi, g\gamma^{-1})]\\
\iff &(\forall[(\psi,g)] \in \mathcal{L}^r_N)\ (\exists f \in \mathrm{GL}_r(F), k \in K(N))\ \psi = f\psi \wedge g\gamma^{-1} = fgk\\
\iff &(\forall[(\psi,g)] \in \mathcal{L}^r_N)\ (\exists k \in K(N))\ g\gamma^{-1} = gk\\
\iff &(\exists k \in K(N))\ \gamma = k^{-1}\\
\iff &\gamma \in K(N).
\end{aligned}$$

The proof for $M^r_{A,K(N)}(\mathbb{C}_\infty)$ is similar. □

We will thus view the above as actions of $\mathrm{GL}_r(A/N)$ on $M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathcal{L}^r_N$, writing elements of $\mathrm{GL}_r(A/N)$ as $[\gamma]$ for $\gamma \in \mathrm{GL}_r\big(\hat{A}\big)$ where necessary.

We now consider this action's effect on their irreducible components:

3.43 Proposition. *The above action of* $\mathrm{GL}_r(A/N)$ *induces an action of* $(A/N)^\times$ *on the irreducible components of* $M^r_{A,K(N)}(\mathbb{C}_\infty)$ *and* $\mathcal{L}^r_N$ *via* $[\gamma] \mapsto [\det \gamma^{-1}]$ *with kernel* $\mathbb{F}_q^\times$ *which leaves the ideal class* $\pi_N(C)$ *of the component* C *unchanged.*

Proof. Under the determinant map of Proposition 3.26, $\mathrm{GL}_r(A/N)$ acts on the components as follows:

$$\det([\gamma][(\psi,g)]) = \det\,[(\psi, g\gamma^{-1})] = [\det g\gamma^{-1}] = [\det g] \cdot [\det \gamma^{-1}].$$

and similarly for $M^r_{A,K(N)}(\mathbb{C}_\infty)$. From this is it easy to see that if two points are in the same component, then they are still in the same component after acting with γ. Now for $\gamma \in \mathrm{GL}_r\big(\hat{A}\big)$, $\det\gamma^{-1} \in \hat{A}^\times$ so that $v_\mathfrak{p}(\det\gamma^{-1}) = 0$ for all prime $\mathfrak{p}$ and hence for an irreducible component C of $\mathcal{L}^r_N$ we have that $\pi_N(\det C \det \gamma^{-1}) = \pi_N(\det C)$. In particular, $\pi_N(\det 1^r_N \det\gamma^{-1}) = 1$, so that $\det 1^r_N \gamma^{-1} \in \operatorname{Im} i_N$.

Finally, γ is in the kernel of this action if and only if $[\det\gamma^{-1}] = [1]$, i.e. $\det\gamma^{-1} = fk$ for some $f \in F^\times$, $k \in \det K(N)$. Since $v_\mathfrak{p}(\det\gamma^{-1}) = v_\mathfrak{p}(k) = 0$ for all prime $\mathfrak{p}$, we have that $v_\mathfrak{p}(f) = 0$ for all prime $\mathfrak{p}$; hence $f \in \mathbb{F}_q^\times$ and so $\det\gamma^{-1} \in \hat{A}^\times \cap \big(\mathbb{F}_q^\times + N\hat{A}\big)$; the converse can be shown easily. Thus $i_N^{-1}(\det 1^r_N\gamma^{-1}) = \mathbb{F}_q^\times \subseteq \mathrm{GL}_r\big(\hat{A}/N\hat{A}\big) \simeq \mathrm{GL}_r(A/N)$. □

So when $\gamma \in \mathrm{GL}_r(A/N)$ acts on $M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathcal{L}^r_N$, it permutes the irreducible components, leaving them fixed if and only if $\det\gamma \in \mathbb{F}_q^\times$.

We now translate this action of $\mathrm{GL}_r(A/N)$ to actions on $M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathcal{L}^r_N$ considered as the set of (equivalence classes of) pairs (Λ, α) of a lattice of rank r with level structure α:

3.44 Proposition. *The action of* $\mathrm{GL}_r(A/N) \simeq \mathrm{GL}_r(\hat{A})/K(N)$ *described as above works on* $M^r_{A,K(N)}(\mathbb{C}_\infty) = \{[(\Lambda, \alpha)]\}$ *and* $\mathcal{L}^r_N = \{(\Lambda, \alpha)\}$ *as follows:*

$$\gamma[(\Lambda, \alpha)] = [(\Lambda, \alpha \circ \gamma^{-1})]; \qquad \gamma(\Lambda, \alpha) = (\Lambda, \alpha \circ \gamma^{-1}).$$

Proof. We will need to revisit the isomorphism Θ in Theorem 3.19:

$$\Theta : \mathrm{GL}_r(F)\backslash\big(\Psi^r \times \mathrm{GL}_r(\mathbb{A}^{fin}_F)/K(N)\big) \simeq \mathcal{L}^r_N.$$

Suppose that $\gamma(\Lambda, \alpha) = \Theta([(\psi, g)])$; here $\gamma[(\psi, g)] = [(\psi, g\hat{\gamma}^{-1})]$ for $\hat{\gamma} \in \mathrm{GL}_r(\hat{A})$ a lift of $\gamma \in \mathrm{GL}_r(A/N)$. Then since $\hat{\gamma}^{-1}\hat{A}^r = \hat{A}^r$,

$$\Lambda' = \psi\big(F^r \cap g\hat{\gamma}^{-1}\hat{A}^r\big) = \psi\big(F^r \cap g\hat{A}^r\big) = \Lambda.$$

To determine α', note that

$$\alpha \circ \subset^{-1} \circ\, g^{-1} = \alpha' \circ \subset^{-1} \circ\, (g\hat{\gamma}^{-1})^{-1} = \alpha' \circ \subset^{-1} \circ\, \hat{\gamma} \circ g^{-1},$$

where $\subset : (N^{-1}/A)^r \hookrightarrow\!\!\!\!\rightarrow N^{-1}\hat{A}^r/\hat{A}^r$ is induced by the inclusion $N^{-1} \subset N^{-1}\hat{A}$. Now for $x = (x_1, \ldots, x_r) \in (N^{-1}/A)^r$,

$$\subset(\gamma(x)) = ((\gamma(x)_1)_{\mathfrak{p}}, \ldots, (\gamma(x)_r)_{\mathfrak{p}}) = (\hat{\gamma}((x)_{\mathfrak{p}})_1, \ldots, \hat{\gamma}((x)_{\mathfrak{p}})_r) = \hat{\gamma}(\subset(x));$$

hence $\subset \circ\, \gamma = \hat{\gamma} \circ \subset$, so that

$$\alpha \circ \subset^{-1} = \alpha' \circ \subset^{-1} \circ\, \hat{\gamma} = \alpha' \circ \gamma \circ \subset^{-1} \iff \alpha' = \alpha \circ \gamma^{-1}. \qquad \square$$

Note that the above action of $\mathrm{GL}_r(A/N)$ on $\mathcal{L}^r_N$ coincides with that defined in the proof of Proposition 3.23.

The action of $\big(\mathbb{A}^{fin}_F\big)^\times$ on $M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathcal{L}^r_N$

3.45 Definition. We define a left action of $\big(\mathbb{A}^{fin}_F\big)^\times$ on $M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathcal{L}^r_N$ by

$$x[(\omega, g)] = [(\omega, gx^{-1})] \quad \text{and} \quad x[(\psi, g)] = [(\psi, gx^{-1})] \quad \text{for } x \in \big(\mathbb{A}^{fin}_F\big)^\times.$$

Since the multiplicative group $\big(\mathbb{A}_F^{fin}\big)^\times$ of invertible finite adeles is abelian, the distinction between a left and a right action is not very important here, but we call it a left action for harmony with the previous section.

3.46 Proposition. *The above action is well defined.*

Proof. If $g \in \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$ then $gx^{-1} \in \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$, since x is invertible. If $[(\omega_1, g_1)] = [(\omega_2, g_2)]$ then there are $f \in \mathrm{GL}_r(F)$ and $k \in K(N)$ such that $\omega_2 = \omega_1 f^{-1}$ and $g_2 = f g_1 k$. Then since x is a scalar, $g_2 x^{-1} = f g_1 x^{-1} k$, so that $[(\omega_1, g_1 x^{-1})] = [(\omega_2, g_2 x^{-1})]$. The proof for $\mathcal{L}_N^r$ is similar. $\square$

3.47 Note that since the above action leaves the components of Ω^r and Ψ^r in the double quotients $\mathrm{GL}_r(F)\backslash\big(\Omega^r \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big) \xrightarrow{\sim} M^r_{A,K(N)}(\mathbb{C}_\infty)$ and $\mathrm{GL}_r(F)\backslash\big(\Psi^r \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big) \xrightarrow{\sim} \mathcal{L}_N^r$ respectively unchanged, and the topology on the quotient $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)$ is discrete, the above actions are rigid analytic automorphisms of the relevant spaces.

3.48 Also note that since $\big(\mathbb{A}_F^{fin}\big)^\times \subset \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$ consists of scalar matrices and thus is in the centre of $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$, the action of $\big(\mathbb{A}_F^{fin}\big)^\times$ commutes with the action of any other subgroup of $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$, and in particular with the actions of $\mathrm{GL}_r\big(\hat{A}\big)$ and $\mathrm{GL}_r(A/N)$ defined in the previous section.

3.49 Proposition. *The kernel of the above action on $\mathcal{L}_N^r$ is $\hat{A}^\times \cap (1 + N\hat{A})$.*

Proof. As in the proof of Proposition 3.42, if $\psi = f\psi$ for $\psi \in \Psi^r$ and $f \in \mathrm{GL}_r(F)$ then $f = 1$. Thus

$$
\begin{aligned}
& x \in \big(\mathbb{A}_F^{fin}\big)^\times \text{ is in the kernel of the action} \\
\iff & (\forall[(\psi, g)] \in \mathcal{L}_N^r)\ [(\psi, g)] = x[(\psi, g)] = [(\psi, gx^{-1})] \\
\iff & (\forall[(\psi, g)] \in \mathcal{L}_N^r)\ (\exists f \in \mathrm{GL}_r(F), k \in K(N))\ \psi = f\psi \wedge gx^{-1} = fgk \\
\iff & (\forall[(\psi, g)] \in \mathcal{L}_N^r)\ (\exists k \in K(N))\ gx^{-1} = gk \\
\iff & (\exists k \in K(N))\ x^{-1}\mathrm{Id}_r = k \\
\iff & x\mathrm{Id}_r \in K(N).
\end{aligned}
$$

For $x\mathrm{Id}_r$ to be in $K(N)$, we must have that $x^r = \det(x\mathrm{Id}_r) \in \hat{A}^\times$, so that $x \in \hat{A}^\times$, and that $(x-1)\mathrm{Id}_r = x\mathrm{Id}_r - \mathrm{Id}_r \in NM_{r\times r}(\hat{A})$, so that $x \equiv_N 1$. $\square$

In other words, $x \in \big(\mathbb{A}_F^{fin}\big)^\times$ is in the kernel of this action if and only if it is an invertible profinite integer with $x - 1 \in N\hat{A}$.

Recall that we denote by $\mathcal{J}(A)$ the set of A-fractional ideals in F.

3.50 Proposition. *There is an abelian group isomorphism*

$$\left(\mathbb{A}_F^{fin}\right)^\times \Big/ \left(\hat{A}^\times \cap (1 + N\hat{A})\right) \xrightarrow{\sim} \mathcal{J}(A) \times (A/N)^\times.$$

Proof. $(A/N)^\times \simeq \prod_{\mathfrak{p}|N} \left(A/\mathfrak{p}^{v_\mathfrak{p}(N)}\right)^\times$, so if we choose a uniformiser $u_\mathfrak{p} \in A_\mathfrak{p}$ for each prime $\mathfrak{p} \mid N$ we can define the forward map

$$\left[x, x \in \left(\mathbb{A}_F^{fin}\right)^\times\right] \mapsto \left(\prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(x)}, \left(\left[x u_\mathfrak{p}^{-v_\mathfrak{p}(x)}\right]\right)_{\mathfrak{p}|N}\right).$$

We show that this map is well defined: if $[x_1] = [x_2]$ for $x_1, x_2 \in \left(\mathbb{A}_F^{fin}\right)^\times$, then $x_2/x_1 \in \hat{A}^\times \cap (1 + N\hat{A})$; hence $v_\mathfrak{p}(x_2/x_1) = 0 \implies v_\mathfrak{p}(x_1) = v_\mathfrak{p}(x_2)$ for each prime $\mathfrak{p}$, so that $\prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}(x_1)} = \prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}(x_2)}$. Moreover, $x_2/x_1 \equiv 1 \pmod{N\hat{A}}$, so that for each prime $\mathfrak{p} \mid N$ we have that $x_1 u_\mathfrak{p}^{-v_\mathfrak{p}(x_1)} \equiv x_2 u_\mathfrak{p}^{-v_\mathfrak{p}(x_2)} \pmod{N\hat{A}}$.

Now we define the inverse map, after choosing a uniformiser $u_\mathfrak{p}$ for *every* prime $\mathfrak{p}$ (although the map defined actually only depends on the choice of uniformiser for $\mathfrak{p} \mid N$). The map is:

$$(J, [n, n \in A]) \mapsto \left[\left(u_\mathfrak{p}^{v_\mathfrak{p}(J)}\right)_{\mathfrak{p}\nmid N} \cup \left(n u_\mathfrak{p}^{v_\mathfrak{p}(J)}\right)_{\mathfrak{p}|N}\right].$$

By composing this inverse map with the described forward map, we see that they are actually inverses, which establishes the isomorphism. □

Note that although the above isomorphism between the quotient of the group $\left(\mathbb{A}_F^{fin}\right)^\times$ by the kernel of its action on $\mathcal{L}_N^r$ and the product $\mathcal{J}(A) \times (A/N)^\times$ is explicit, it is not canonical since it depends on the choice of uniformisers $u_\mathfrak{p}$ for $\mathfrak{p} \mid N$. The following, although a weaker result, is canonical:

3.51 Proposition. *There is a short exact sequence*

$$0 \hookrightarrow (A/N)^\times \xrightarrow{x_N} \left(\mathbb{A}_F^{fin}\right)^\times \Big/ \left(\hat{A}^\times \cap (1 + N\hat{A})\right) \xrightarrow{J} \mathcal{J}(A) \twoheadrightarrow 0,$$

the maps x_N and J given by

$$(A/N)^\times \ni [n, n \in A] \xmapsto{x_N} [(1)_{\mathfrak{p}\nmid N} \cup (n)_{\mathfrak{p}|N}]$$

and

$$[x] \xmapsto{J} \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(x)} \in \mathcal{J}(A).$$

Proof. These maps are extracted from the proof of Proposition 3.50. □

Note that $J(x) = x\hat{A} \cap F$ is the unique fractional ideal in F such that $x\hat{A} = J(x)\hat{A}$.

There is a related short exact sequence; for its proof, keep in mind the identification

$$\left(\mathbb{A}_F^{fin}\right)^\times \supset F^\times (A/N)^\times \simeq F^\times \times (A/N)^\times / (F^\times \cap (A/N)^\times) \simeq (F^\times/\mathbb{F}_q^\times) \times (A/N)^\times.$$

3.52 Proposition. *There is a short exact sequence*

$$0 \hookrightarrow F^\times (A/N)^\times \xhookrightarrow{x_{N,F}} \left(\mathbb{A}_F^{fin}\right)^\times \Big/ \left(\hat{A}^\times \cap (1 + N\hat{A})\right) \xrightarrow{[J]} \twoheadrightarrow \mathrm{Cl}(F) \twoheadrightarrow 0,$$

with the maps $x_{N,F}$ and $[J]$ given by

$$F^\times (A/N)^\times \ni ([f], [n, n \in A]) \xmapsto{x_{N,F}} [(f)_{\mathfrak{p}\nmid N} \cup (fn)_{\mathfrak{p}|N}]$$

and

$$[x] \xmapsto{[J]} \left[\prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x)}\right]_{\mathrm{Cl}(F)}.$$

Proof. The proof is similar to that of Proposition 3.51. □

We now investigate this action of $\left(\mathbb{A}_F^{fin}\right)^\times$ on $\mathcal{L}_N^r$ considered as the set of pairs (Λ, α) of a lattice Λ with level N structure α. But first, an observation on the action of $x \in \left(\mathbb{A}_F^{fin}\right)^\times$ on ideal quotients:

3.53 Lemma. *If $I \in \mathcal{J}(A)$ is a fractional ideal and $x \in \left(\mathbb{A}_F^{fin}\right)^\times$ with corresponding $J = J(x)$, then there is a natural A/N-module isomorphism $x^{-1} : N^{-1}I/I \hookrightarrow\!\!\!\!\rightarrow N^{-1}J^{-1}I/J^{-1}I$ given by*

$$\begin{tikzcd}
N^{-1}I/I \arrow[r, hook, two heads, dotted, "x^{-1}"] \arrow[d, hook, two heads, "\subset"'] & N^{-1}J^{-1}I/J^{-1}I = N^{-1}(x^{-1}I\hat{A} \cap F)/(x^{-1}I\hat{A} \cap F) \arrow[d, hook, two heads, "\subset"] \\
N^{-1}I\hat{A}/I\hat{A} \arrow[r, hook, two heads, "x^{-1}"'] & N^{-1}x^{-1}I\hat{A}/x^{-1}I\hat{A}
\end{tikzcd}$$

3.54 Proposition. *If $x(\Lambda, \alpha) = (\Lambda', \alpha')$ for $x \in \left(\mathbb{A}_F^{fin}\right)^\times$ with fractional ideal $J = J(x) \in \mathcal{J}(A)$, then $\Lambda' = J^{-1}\Lambda$ and α' is given as the composite*

$$\left(N^{-1}/A\right)^r \xhookrightarrow{\alpha} N^{-1}\Lambda/\Lambda \xhookrightarrow{x^{-1}} N^{-1}J^{-1}\Lambda/J^{-1}\Lambda.$$

Proof. Let $\Lambda = I_1\psi_1 + \cdots + I_r\psi_r$ for fractional ideals I_i and $\psi \in \Psi^r$. Then choosing the rows g_i' of $g' = (g_1', \ldots, g_r')^T$ such that $F \cap g_i\hat{A}^r = I_i$ for each i, we have that $\psi\big(F^r \cap g'\hat{A}^r\big) = \psi \cdot (I_1, \ldots, I_r)^T = \Lambda$. Then since $\mathrm{GL}_r\big(\hat{A}\big)$ acts transitively on the set of level structures for any given lattice, for a suitable $\gamma \in \mathrm{GL}_r\big(\hat{A}\big)$ we will have that $(\Lambda, \alpha) = \Theta([(\psi, g)])$ for $g = g'\gamma$, since $g\hat{A}^r = g'\gamma\hat{A}^r = g'\hat{A}^r$.

Now $x[(\psi, g)] = [(\psi, gx^{-1})]$, and so

$$\begin{aligned}
\Lambda' &= \psi\big(F^r \cap (gx^{-1})\hat{A}^r\big) = \psi\big(F^r \cap g(x^{-1}\hat{A})^r\big) \\
&= \psi\big((J^{-1}F)^r \cap g(J^{-1}\hat{A})^r\big) = \psi\big(J^{-1}\big(F^r \cap g\hat{A}^r\big)\big) \\
&= J^{-1}\psi\big(F^r \cap g\hat{A}^r\big) = J(x)^{-1}\Lambda.
\end{aligned}$$

Comparing the two corresponding versions of Diagram 3.20 for α and α', note that ψ is common in both. Hence for $\Lambda = I_1\psi_1 + \cdots + I_r\psi_r$ we have the following commutative diagram, making use of Lemma 3.53:

$$\begin{tikzcd}
(N^{-1}/A)^r \arrow[r, hook, two heads, "\alpha"] \arrow[dr, dotted, two heads, "\alpha'"'] & N^{-1}\Lambda/\Lambda \arrow[r, hook, two heads, "\psi^{-1}"] \arrow[d, dotted, hook, two heads, "x^{-1}"'] & \bigoplus_{i=1}^r N^{-1}I_i/I_i \arrow[d, hook, two heads, "x^{-1}"] \\
& N^{-1}J^{-1}\Lambda/J^{-1}\Lambda & \bigoplus_{i=1}^r N^{-1}J^{-1}I_i/J^{-1}I_i \arrow[l, hook, two heads, "\psi"]
\end{tikzcd}$$

the dotted arrows defined so as to make the diagram commute. $\square$

Hence the action of $\big(\mathbb{A}_F^{fin}\big)^\times$ on $\mathcal{L}_N^r$ induces a rigid analytic action of the set of fractional ideals on $\mathcal{L}^r$ by $\Lambda \overset{J}{\mapsto} J^{-1}\Lambda$.

4 Lattices with metric structure

A metric on the space of lattices

Metric definitions

The space of all prelattices can be equipped with the structure of a metric space using the associated exponential functions:

4.1 Definition. The metric d_V is defined on the space V of all prelattices as follows:

$$\mathrm{d}_V(\Lambda_1, \Lambda_2) = \sup_{|z| \leq 1} |e_{\Lambda_1}(z) - e_{\Lambda_2}(z)| \quad \text{for } \Lambda_1, \Lambda_2 \in V.$$

If in addition Λ_1, Λ_2 are lattices, we may use the notation $\mathrm{d}_{\mathcal{L}}$ instead.

4.2 Proposition. *The above is a metric.*

Proof. Firstly, $\mathrm{d}_V(\Lambda_1, \Lambda_2) = \mathrm{d}_V(\Lambda_2, \Lambda_1) \geq 0$ for all $\Lambda_1, \Lambda_2 \in V$.

Secondly, for $\Lambda_1, \Lambda_2, \Lambda_3 \in V$,

$$\begin{aligned} \mathrm{d}_V(\Lambda_1, \Lambda_2) + \mathrm{d}_V(\Lambda_2, \Lambda_3) &= \sup_{|z| \leq 1} |e_{\Lambda_1}(z) - e_{\Lambda_2}(z)| + \sup_{|z| \leq 1} |e_{\Lambda_2}(z) - e_{\Lambda_3}(z)| \\ &\geq \sup_{|z| \leq 1} (|e_{\Lambda_1}(z) - e_{\Lambda_2}(z)| + |e_{\Lambda_2}(z) - e_{\Lambda_3}(z)|) \\ &\geq \sup_{|z| \leq 1} |e_{\Lambda_1}(z) - e_{\Lambda_2}(z) + e_{\Lambda_2}(z) - e_{\Lambda_3}(z)| \\ &= \mathrm{d}_V(\Lambda_1, \Lambda_3). \end{aligned}$$

Finally, if $\mathrm{d}_V(\Lambda_1, \Lambda_2) = 0$ then $e_{\Lambda_1}(z) = e_{\Lambda_2}(z)$ for all $z \in \mathbb{C}_\infty$ with $|z| \leq 1$. Then since e_{Λ_1} and e_{Λ_2} have power series expansions convergent on all of $\mathbb{C}_\infty$, they are equal on $\mathbb{C}_\infty$ and thus

$$\Lambda_1 = \{z \in \mathbb{C}_\infty \mid e_{\Lambda_1}(z) = 0\} = \{z \in \mathbb{C}_\infty \mid e_{\Lambda_2}(z) = 0\} = \Lambda_2. \qquad \square$$

The restriction to $|z| \leq 1$ in the above metric definition is largely superfluous in that the same local topology is generated if we replace it with $|z| \leq R$, as shown by the following proposition, the proof of which will be postponed until Page 71 in Chapter 5:

4.3 Proposition. *Let $R > 0$ and let Λ and Λ' be lattices of rank $\leq r$, with Λ' being variable. If $\Lambda' \to \Lambda$, i.e.*

$$\mathrm{d}_{\mathcal{L}}(\Lambda', \Lambda) = \sup_{|z|\leq 1} |e_{\Lambda'}(z) - e_{\Lambda}(z)| \to 0,$$

then

$$\sup_{|z|\leq R} |e_{\Lambda'}(z) - e_{\Lambda}(z)| \to 0.$$

With this metric, we can consider the 'size' of a prelattice to be its distance to the zero lattice, which by an abuse of notation we denote as 0. As it turns out, prelattices all of whose nonzero elements are large are 'small':

4.4 Proposition. *If a prelattice Λ has $\min'_{\lambda\in\Lambda}|\lambda| = R > 1$, then $\mathrm{d}_V(\Lambda, 0) \leq R^{1-q}$.*

Proof. Apply Proposition 2.18. □

4.5 Corollary. *If $\min'_{\lambda\in\Lambda}|\lambda| \to \infty$, then $\Lambda \to 0$.*

More particularly, elements of a lattice which are 'large' make a 'small' difference in the topology induced by this metric as shown by Proposition 4.7. But first, a lemma:

4.6 Lemma. *For any nonzero fractional ideal $J \in \mathcal{J}(A)$, there is a bound $B_J > 0$ such that for every $f \in F$ there is a $n \in J$ with $|f - n| < B_J$.*

Proof. Let $T \in A$ with $|T| > 1$. Then $\mathbb{F}_q[T]$ is a principal ideal domain and a subring of A. Thus J, as a torsion-free $\mathbb{F}_q[T]$-module, is free:

$$J = j_1\mathbb{F}_q[T] + \cdots + j_m\mathbb{F}_q[T] \quad \text{for } \mathbb{F}_q(T) - \text{independent } j_i \in J.$$

The field F can be written similarly $F = j_1\mathbb{F}_q(T) + \cdots + j_m\mathbb{F}_q(T)$.

Now $\mathbb{F}_q[T]$ is a Euclidean domain, and so for any $f' \in \mathbb{F}_q(T)$ there is a $t \in \mathbb{F}_q[T]$ with $|f' - t| < 1$. Thus for any $f = j_1 f'_1 + \cdots + j_m f'_m \in F$ with $f'_i \in \mathbb{F}_q(T)$, we can find corresponding $t_i \in \mathbb{F}_q[T]$ with $|f'_i - t_i| < 1$ for each i, so that with $n = j_1 t_1 + \cdots j_m t_m \in I$ we have

$$|f - n| \leq \max_{i=1}^{m} (|j_i| \cdot |f'_i - t_i|) < \max_{i=1}^{m} |j_i| =: B_J. \qquad \square$$

4.7 Proposition. *Let Λ be a fixed lattice of rank r. Then for variable $\omega \in \mathbb{C}_\infty$ with $\mathrm{d}(\omega, \Lambda) = \min_{\lambda\in\Lambda}|\omega - \lambda| \to \infty$, we have that $\Lambda + A\omega \to \Lambda$ with respect to d_V.*

Proof. Λ is a sublattice of $\Lambda + A\omega$, so from Proposition 2.15, we have that

$$e_{\Lambda+A\omega}(z) = e_{e_\Lambda(\Lambda+A\omega)}(e_\Lambda(z)) = e_{e_\Lambda(A\omega)}(e_\Lambda(z)).$$

Now let $R_1 = \mathrm{d}(\omega, \Lambda) = \min_{\lambda\in\Lambda}|\omega - \lambda|$ be large, so that $R_2 = |e_\Lambda(\omega)|$ is large by Proposition 2.16. Then for any nonzero $a \in A$,

$$|e_\Lambda(a\omega)| = |a||e_\Lambda(\omega)| \prod_{\lambda\in a^{-1}\Lambda/\Lambda}{}' \frac{|e_\Lambda(\omega) - e_\Lambda(\lambda)|}{|e_\Lambda(\lambda)|}.$$

Now by Lemma 4.6, there is a bounded set of representatives of F/A; hence since e_Λ is entire the $e_\Lambda(\lambda)$ above are bounded independently of a, say by $L > 0$. Hence for large enough $R_2 > L$ independent of a,

$$|e_\Lambda(a\omega)| = |a|\frac{|e_\Lambda(\omega)|^{|a|^r}}{\prod'_{\lambda\in\Lambda}|e_\Lambda(\lambda)|} \geq |a|\left[\frac{|e_\Lambda(\omega)|}{L}\right]^{|a|^r} \geq R_2,$$

and so $\mathrm{d}(A\omega - \{0\}, \Lambda)$ is large; thus $R_3 = \min'_{\lambda\in e_\Lambda(A\omega)}|\lambda|$ is large.

Now let $\sup_{|z|\leq 1}|e_\Lambda(z)| = m$ which is independent of ω; since R_3 is large, we can also suppose that $R_3 > m$. Then

$$\begin{aligned}
\mathrm{d}_{\mathcal{L}}(\Lambda + A\omega, \Lambda) &= \sup_{|z|\leq 1}|e_{e_\Lambda(A\omega)}(e_\Lambda(z)) - e_\Lambda(z)| \\
&\leq \sup_{|z|\leq m}|e_{e_\Lambda(A\omega)}(z) - z| \\
&\leq \sup_{|z|\leq m}|z|^q R_3^{1-q} = m^q R_3^{1-q} \quad \text{by Proposition 2.18} \\
&\to 0 \quad \text{as } R_3 \to \infty.
\end{aligned}$$

Thus $\Lambda + A\omega \to \Lambda$ as desired. □

Metric completeness

We have just seen an example where a variable lattice of rank $r+1$ tended to a lattice of rank r. In general, we have the result of Corollary 4.9; but first, a proposition:

4.8 Proposition. *$\mathcal{L}^{\leq r}$ is complete.*

Proof. Let (Λ_n) be a Cauchy sequence in $\mathcal{L}^{\leq r}$, which we will show converges to a lattice in $\mathcal{L}^{\leq r}$. Also let the Λ_n have associated Drinfeld modules ϕ^n respectively. Since the Λ_n form a Cauchy sequence, the $\mathbb{F}_q$-linear analytic functions e_{Λ_n} form a Cauchy sequence on the unit disk under the sup-norm by the definition of d_V, and hence converge to an $\mathbb{F}_q$-linear analytic function e on the unit disk by [FP|RAG, Section 2.2]. Since each of these functions have linear coefficient equal to 1, we have that in a neighbourhood of zero the inverse functions $e_{\Lambda_n}^{-1}$ converge uniformly to e^{-1}.

Now let $a \in A$. Then for X in a neighbourhood of zero, we have that

$$\phi_a^n(X) = e_{\Lambda_n}(a \cdot e_{\Lambda_n}^{-1}(X)) \to e(a \cdot e^{-1}(X)) \quad \text{uniformly} .$$

But by Proposition 3.1, $\phi_a^n(X)$ is an $\mathbb{F}_q$-linear polynomial of degree at most $q^{r \deg a}$, and hence $\phi_a(X) := e(a \cdot e^{-1}(X))$ is too. Moreover, it is easy to see that $\phi_{ab} = \phi_a \circ \phi_b$ and $\phi_{a+b} = \phi_a + \phi_b$ for $a, b \in A$, so that ϕ is a ring homomorphism into $\mathbb{C}_\infty\{\tau\}$, and that the linear coefficient of ϕ_a is a. So by [Go|Bas, Section 4.5, p. 70-71], ϕ is a Drinfeld module of rank d for some nonnegative integer d; that $d \leq r$ is easy to see.

So ϕ, being a Drinfeld module of rank d, has a corresponding lattice Λ of rank d and exponential function e_Λ, and $\phi = \phi^\Lambda$. Now for each $a \in A$, $e(a \cdot e^{-1}(X)) = \phi_a(X) = \phi_a^\Lambda(X) = e_\Lambda(a \cdot e_\Lambda^{-1}(X))$ for small X, so that the $\mathbb{F}_q$-linear function $i = e_\Lambda^{-1} \circ e$, analytic in a neighbourhood of zero with linear coefficient equal to 1, satisfies $i(aX) = ai(X)$ for all $a \in A$. From this, by considering the Taylor expansion of i we see that it is the identity function, so that $e_\Lambda = e$ is the limit of the functions e_{Λ_n}, and so $\Lambda_n \to \Lambda$ as desired. □

4.9 Corollary. *If $(\Lambda_n)_{n=1}^\infty$ is a sequence of lattices of rank $\leq R$ and Λ a lattice of rank r such that $\Lambda_n \to \Lambda$, then $r \leq R$.*

Proof. The lattices Λ_n are elements of $\mathcal{L}^{\leq R}$, which is complete by Proposition 4.8. Since the Λ_n form a Cauchy sequence, they hence converge to a lattice $\Lambda_R \in \mathcal{L}^{\leq R}$. But then $\Lambda_R = \Lambda$, and so Λ is of rank at most R. □

4.10 Proposition. *$\mathcal{L}^r$ is a dense subset of $\mathcal{L}^{\leq r}$.*

Proof. Let Λ be a lattice of rank $s \leq r$, let $f \in F$ with $|f| > 1$, and let $\omega_{s+1}, \ldots, \omega_r \in \mathbb{C}_\infty$ be F_∞-linearly independent with Λ, in other words $F_\infty \otimes \Lambda + F_\infty \omega_{s+1} + \cdots + F_\infty \omega_r$ has dimension r as an F_∞-vector space. Finally let $\Lambda_n = \Lambda + Af^n\omega_{s+1} + \cdots + Af^n\omega_r$ be lattices of rank r for $n \in \mathbb{N}$; we will show that $\Lambda_n \to \Lambda$.*

*If $s = r$, then $\Lambda_n = \Lambda$ for each n.

Similarly to Proposition 4.7, we consider Λ as a sublattice of Λ_n, so that $e_{\Lambda_n}(z) = e_{e_\Lambda(\Lambda_n)}(e_\Lambda(z))$. Now $e_\Lambda(\Lambda_n) = e_\Lambda(f^n(A\omega_{s+1} + \cdots + A\omega_r))$, which we show goes to the zero lattice as $n \to \infty$. Let

$$R_1 = \mathrm{d}(A\omega_{s+1} + \cdots + A\omega_r, F_\infty\Lambda) = \inf_{\substack{s+1\leq \ell\leq r\\ x\in F_\infty\Lambda}} |\omega_\ell - x|,$$

which is positive since $A\omega_{s+1} + \cdots + A\omega_r$ is strongly discrete and the ω_ℓ are F_∞-linearly independent with Λ. Thus as $n \to \infty$,

$$\begin{aligned}
&\mathrm{d}(f^n(A\omega_{s+1} + \cdots + A\omega_r) - \{0\}, \Lambda)\\
\geq &\,\mathrm{d}(f^n(A\omega_{s+1} + \cdots + A\omega_r) - \{0\}, F_\infty\Lambda)\\
= &\,|f|^n \,\mathrm{d}(A\omega_{s+1} + \cdots + A\omega_r - \{0\}, F_\infty\Lambda)\\
\to &\infty,
\end{aligned}$$

so $R_n = \min'_{\lambda\in e_\Lambda(\Lambda_n)}|\lambda| \to \infty$ as $n \to \infty$ by Proposition 2.16.

Finally, let $\sup\limits_{|z|\leq 1}|e_\Lambda(z)| = m$; then

$$\begin{aligned}
\mathrm{d}_{\mathcal{L}}(\Lambda_n, \Lambda) &= \sup_{|z|\leq 1}|e_{e_\Lambda(\Lambda_n)}(e_\Lambda(z)) - e_\Lambda(z)|\\
&\leq \sup_{|z|\leq m}|e_{e_\Lambda(\Lambda_n)}(z) - z|\\
&\leq \sup_{|z|\leq m}|z|^q R_n^{1-q} = m^q R_n^{1-q} \quad \text{by Proposition 2.18}\\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

so that $\Lambda_n \to \Lambda$ as desired. □

By Proposition 4.7, we see that the space $\mathcal{L}^r$ of lattices of rank r is not complete with respect to $\mathrm{d}_{\mathcal{L}}$. However, combining the two previous propositions we have:

4.11 Theorem. *The space $\mathcal{L}^{\leq r}$ is the completion of $\mathcal{L}^r$.*

Proof. By Propositions 4.8 and 4.10, $\mathcal{L}^{\leq r}$ is complete with dense subset $\mathcal{L}^r$. □

We also note the following trivial proposition without proof for explicitness:

4.12 Proposition.

$$\mathcal{L}^{\leq r} = \bigsqcup_{0\leq s\leq r} \mathcal{L}^s.$$

4.13 Definition. In the above decomposition of $\mathcal{L}^{\leq r}$ into a disjoint union of $\mathcal{L}^s$ for $0 \leq s \leq r$, each of the uniands[†] $\mathcal{L}^s$ is called a *stratum* of *dimension* s and *codimension* $r - s$; if $s < r$, the $\mathcal{L}^s$ are also called *boundary strata*.

Topology of the irreducible components

In this subsection we let C denote an irreducible component of $\mathcal{L}^r \simeq \mathcal{L}^r_A$, these components defined by the values of $\pi(\Lambda) \in \mathrm{Cl}(F)$ for $\Lambda \in \mathcal{L}^r$. (Recall that π is defined in Proposition 3.30, with N being irrelevant due to Paragraph 3.34.)

So we see that the space $\mathcal{L}^r$ has boundary $\partial\mathcal{L}^r = \partial\mathcal{L}^{\leq r} = \mathcal{L}^{\leq r-1}$ in the complete metric space $\mathcal{L}^{\leq r}$ consisting of the Λ with Λ of rank strictly less than r. We will see that each of the irreducible components of $\mathcal{L}^r$ possesses the same boundary.

First, let us look closer at $\mathcal{L}^{\leq r-1}$:

4.14 Proposition. *$\mathcal{L}^{\leq r-1}$ is a closed subset of $\mathcal{L}^{\leq r}$, so $\mathcal{L}^r = \mathcal{L}^{\leq r} - \mathcal{L}^{\leq r-1}$ is open.*

Proof. Let $(\Lambda_n)_{n=0}^{\infty}$ be a Cauchy sequence in $\mathcal{L}^{\leq r-1}$, so that each Λ_n has rank strictly less than r. Then since $\mathcal{L}^{\leq r}$ is complete, this sequence has a limit $\Lambda \in \mathcal{L}^{\leq r}$. Hence by Corollary 4.9 we have that Λ has rank strictly less than r, so that $\Lambda \in \mathcal{L}^{\leq r-1}$ as desired. $\square$

4.15 Proposition. *C is a dense subset of $C \cup \mathcal{L}^{\leq r-1}$.*

Proof. Let $\Lambda \in \mathcal{L}^{\leq r-1}$ with $\Lambda = I_1\psi_1 + \cdots + I_s\psi_s$ of rank $s < r$, let $f \in F$ with $|f| > 1$, and let $\psi_{s+1}, \ldots, \psi_r \in \mathbb{C}_\infty$ be F_∞-linearly independent with Λ. Also let $I \in \mathcal{J}(A)$ such that $[I_1 \cdots I_s I]_{\mathrm{Cl}(F)} = \pi(C)$, let

$$\Lambda' = A\psi_{s+1} + \cdots + A\psi_{r-1} + I\psi_r$$

be a lattice of rank $r - s$ and let $\Lambda_n = \Lambda + f^n\Lambda'$ be lattices of rank r for $n \in \mathbb{N}$. Then each Λ_n has

$$\pi(\Lambda_n) = [I_1 \cdots I_s A \cdots AI]_{\mathrm{Cl}(F)} = \pi(C),$$

so that each Λ_n also lies in the component C, and similarly to the proof of Proposition 4.10 we have that $\Lambda_n \to \Lambda$. $\square$

4.16 Proposition. *C is open in $\mathcal{L}^{\leq r}$.*

[†]Uniands are to unions as summands are to sums.

Proof. Let $\Lambda \in C$. We will find an open neighbourhood $U \subseteq \mathcal{L}^{\leq r}$ of Λ such that $U \subseteq C$, or equivalently that each $\Lambda' \in U$ has $\pi(\Lambda') = \pi(\Lambda)$; by Proposition 3.32 and Paragraph 2.3, it is sufficient to establish an A-module isomorphism between Λ and Λ' for $\Lambda' \in U$.

For each $z \in \mathbb{C}_\infty$ and $Q > 0$ we will denote by

$$\mathrm{B}[z, Q] = \{y \in \mathbb{C}_\infty \mid |z - y| \leq Q\} \text{ and } \mathrm{B}(z, Q) = \{y \in \mathbb{C}_\infty \mid |z - y| < Q\}$$

the closed and open balls around z of radius Q respectively.

As in Proposition 4.3, for any $R > 0$ we have that

$$\Lambda' \to \Lambda \implies \sup_{|z| \leq R} |e_{\Lambda'}(z) - e_\Lambda(z)| \to 0.$$

By Lemma 4.6, there is a bound $M > 1$ (which we fix) such that for every $f \in F$ there is an $x \in f + A$ with $|x| < M$. We let R be fixed and large enough that there are $\lambda_1, \ldots, \lambda_r \in \Lambda \cap \mathrm{B}[0, R/M]$, together with fractional ideals $I_1, \ldots, I_r$ each of which contains A, such that $\Lambda = I_1\lambda_1 + \cdots + I_r\lambda_r$, and define the ball $\mathrm{B}_R = \mathrm{B}[0, R] = \{z \in \mathbb{C}_\infty \mid |z| \leq R\}$.

Since $\mathrm{B}_R \cap \Lambda$ is finite, so is $F' = \{\lambda_1/\lambda_2 \mid \lambda_1, \lambda_2 \in \Lambda \cap \mathrm{B}_R\} \cap F$; so we can choose a $T \in A$ with $|T| > 1$ such that $T \cdot F' = \{Tf \mid f \in F'\} \subset A$.

Also, note that F_∞ is complete with respect to $|\cdot|$ and that $F_\infty\Lambda \subset \mathbb{C}_\infty$ is an F_∞-vector space of dimension r and basis $(\lambda_1, \ldots, \lambda_r)$, with an induced norm $|\cdot|$ inherited from $\mathbb{C}_\infty$, so by [Ne|ANT, Proposition 4.9, p. 132] all norms are equivalent; in particular, there is a $\rho > 0$ such that

$$|f_1\lambda_1 + \cdots + f_r\lambda_r| \geq \rho(|f_1| + \cdots + |f_r|) \quad \text{for all } f_1, \ldots, f_r \in F_\infty.$$

We also choose an $\epsilon > 0$ such that

$$\epsilon < \min\left\{ \min_{\lambda \in \Lambda}{}' \frac{|\lambda|}{|T|M}, \min_{\lambda \in \Lambda}{}' \frac{\rho|\lambda|}{R}, \rho \right\}.$$

We will now construct an A-module bijection between Λ and Λ' for any Λ' such that $\sup_{|z| \leq R} |e_{\Lambda'}(z) - e_\Lambda(z)| < \epsilon$. In other words, the open neighbourhood U of Λ is an 'open ball' around Λ of radius ϵ, which we will show lies inside C.

Define the balls $\lambda_\epsilon = \mathrm{B}(\lambda, \epsilon) = \{z \in \mathbb{C}_\infty \mid |z - \lambda| < \epsilon\}$ for $\lambda \in \Lambda \cup \Lambda'$. Note that the balls λ_ϵ for $\lambda \in \Lambda$ are disjoint, since if distinct $\lambda_1, \lambda_2 \in \Lambda$ then

$$\max\{|z - \lambda_1|, |z - \lambda_2|\} \geq |z - \lambda_1 - z + \lambda_2| = |\lambda_2 - \lambda_1| > |T|M\epsilon > \epsilon.$$

Moreover for $z \in \lambda_\epsilon$,

$$|e_\Lambda(z)| = |e_\Lambda(z-\lambda)| = |z-\lambda| \prod_{\lambda_0 \in \Lambda}{}' \left|1 - \frac{z-\lambda}{\lambda_0}\right| = |z-\lambda| < \epsilon.$$

So $z \in \lambda_\epsilon \implies |e_\Lambda(z)| < \epsilon$, and conversely if $z \notin \lambda_\epsilon$ for all $\lambda \in \Lambda$ then

$$\left|\frac{1}{e_\Lambda(z)}\right| \le \left|\sum_{\lambda \in \Lambda} \frac{1}{z-\lambda}\right| \le \max_{\lambda \in \Lambda} \frac{1}{|z-\lambda|} \le \frac{1}{\epsilon} \implies |e_\Lambda(z)| \ge \epsilon.$$

Now for any $\lambda' \in \Lambda' \cap \mathrm{B}_R$ we have that $|e_\Lambda(\lambda')| = |e_\Lambda(\lambda') - e_{\Lambda'}(\lambda')| < \epsilon$, so $\lambda' \in \lambda_\epsilon$ for a unique $\lambda \in \Lambda$, since the λ_ϵ are disjoint. Thus

$$|\lambda'| = |\lambda| > |T|M\epsilon > \epsilon \quad \text{for all } \lambda' \in \Lambda' - \{0\}.$$

(In particular, $\min'_{\lambda' \in \Lambda'}|\lambda'| = \min'_{\lambda \in \Lambda}|\lambda|$.)

Hence for any $\lambda \in \Lambda \cap \mathrm{B}_R$ we have that $|e_{\Lambda'}(\lambda)| = |e_{\Lambda'}(\lambda) - e_\Lambda(\lambda)| < \epsilon$, so that by similar reasoning to before we have that $\lambda \in \lambda'_\epsilon$ for a unique $\lambda' \in \Lambda'$, the balls λ'_ϵ also being disjoint. So for any $\lambda \in \Lambda \cap \mathrm{B}_R$ we denote this value of λ' by $f(\lambda) \in \Lambda' \cap \lambda_\epsilon$.

With $\lambda_1, \ldots, \lambda_r \in \Lambda$ and $I_1, \ldots, I_r \in \mathcal{J}(A)$ as chosen before, we define

$$\overline{f} : \Lambda \to \Lambda', \qquad l_1\lambda_1 + \cdots + l_r\lambda_r \mapsto l_1 f(\lambda_1) + \cdots + l_r f(\lambda_r) \quad \text{for } l_i \in I_i.$$

To show that this map is well defined, it is enough to show that $l_i f(\lambda_i) \in \Lambda'$ for each $l_i \in I_i$. Obviously each $f(\lambda_i) \in \Lambda'$, so taking l_i modulo A we may without loss of generality assume that $|l_i| < M$. Now $|l_i\lambda_i| < M \cdot R/M = R$, so that $l_i\lambda_i \in \Lambda \cap \mathrm{B}_R$; we now show that $l_i f(\lambda_i) = f(l_i\lambda_i) \in \Lambda'$, noting that $l_i = l_i\lambda_i/\lambda_i \in F'$. Thus $Tl_i \in A$, so that $Tf(l_i\lambda_i) - Tl_i f(\lambda_i) \in \Lambda'$. Now

$$|Tf(l_i\lambda_i) - Tl_i f(\lambda_i)| \le |T| \max\{|f(l_i\lambda_i) - l_i\lambda_i|, |l_i(f(\lambda_i) - \lambda_i)|\} = |T|M\epsilon;$$

thus $Tf(l_i\lambda_i) - Tl_i f(\lambda_i) = 0$, so $f(l_i\lambda_i) = l_i f(\lambda_i)$ as desired.

We now show that $\overline{f}(\lambda) = f(\lambda)$ for $\lambda = l_1\lambda_1 + \cdots + l_r\lambda_r \in \Lambda \cap \mathrm{B}_R$. Indeed,

$$R > |l_1\lambda_1 + \cdots + l_r\lambda_r| \ge \rho \cdot (|l_1| + \cdots + |l_r|) \implies |l_1| + \cdots + |l_r| < R/\rho;$$

thus

$$\begin{aligned}
\left|\lambda - \overline{f}(\lambda)\right| &\leq |l_1(\lambda_1 - f(\lambda_1))| + \cdots + |l_r(\lambda_r - f(\lambda_r))| \\
&< \epsilon \cdot (|l_1| + \cdots + |l_r|) < \frac{\epsilon R}{\rho} \\
&< \min_{\lambda\in\Lambda}{}'|\lambda| = \min_{\lambda'\in\Lambda'}{}'|\lambda'| \\
\implies \left|f(\lambda) - \overline{f}(\lambda)\right| &\leq \max\left\{|f(\lambda) - \lambda|, \left|\lambda - \overline{f}(\lambda)\right|\right\} < \max\left\{\min_{\lambda'\in\Lambda'}{}'|\lambda'|, \epsilon\right\} \\
&= \min_{\lambda'\in\Lambda'}{}'|\lambda'|.
\end{aligned}$$

But $\overline{f}(\lambda) \in \Lambda'$, so $f(\lambda) - \overline{f}(\lambda) \in \Lambda'$, so we must have that $\overline{f}(\lambda) = f(\lambda)$.

To show that $\overline{f}$ is injective, i.e. that the $f(\lambda_i)$ are A-linearly independent, suppose that $l_1 f(\lambda_1) + \cdots + l_r f(\lambda_r) = 0$ where the $l_i \in I_i$. Then

$$\begin{aligned}
\rho \cdot (|l_1| + \cdots + |l_r|) &\leq |l_1\lambda_1 + \cdots + l_r\lambda_r| \\
&= |l_1(\lambda_1 - f(\lambda_1)) + \cdots + l_r(\lambda_r - f(\lambda_r))| \\
&\leq \max_{i=1}^{r} |l_i| \cdot |\lambda_i - f(\lambda_i)| \\
&< \epsilon \cdot (|l_i| + \cdots + |l_r|);
\end{aligned}$$

thus all the l_i are zero since $\epsilon < \rho$.

To show that $\overline{f}$ is surjective, since $I_1 f(\lambda_1) + \cdots + I_r f(\lambda_r) \subseteq \Lambda'$ is of rank r, every $\lambda' \in \Lambda'$ is of the form $\lambda' = l_1 f(\lambda_1) + \cdots + l_r f(\lambda_r)$ for $l_i \in F$. Taking the l_i modulo the I_i respectively, we may without loss of generality assume that each $|l_i| < M$. Thus $|\lambda'| = |l_1 f(\lambda_1) + \cdots + l_r f(\lambda_r)| < M \cdot R/M = R$, so that $\lambda' \in \Lambda' \cap \mathrm{B}_R$. Then $|e_\Lambda(\lambda')| = |e_\Lambda(\lambda') - e_{\Lambda'}(\lambda')| < \epsilon$, so that there is a $\lambda \in \Lambda \cap \mathrm{B}_R$ with $|\lambda - \lambda'| < \epsilon$; thus $\lambda' = f(\lambda) = \overline{f}(\lambda)$ as desired.

Thus $\overline{f} : \Lambda \to \Lambda'$ is an A-module isomorphism as desired. □

4.17 Corollary. *$C \cup \mathcal{L}^{\leq r-1}$ is closed in $\mathcal{L}^{\leq r}$.*

Proof. If $\mathcal{D}$ denotes the set of irreducible components of $\mathcal{L}^r$, then

$$\mathcal{L}^{\leq r} - \left(C \cup \mathcal{L}^{\leq r-1}\right) = \mathcal{L}^r - C = \bigcup_{C'\in\mathcal{D}-\{C\}} C' \quad \text{is open.}$$ □

4.18 Theorem. *$C \cup \mathcal{L}^{\leq r-1}$ is the completion of C, i.e. the closure of C in $\mathcal{L}^{\leq r}$.*

Proof. By Proposition 4.15 and Corollary 4.17, C is dense in $C \cup \mathcal{L}^{\leq r-1}$, which is closed in $\mathcal{L}^{\leq r}$ and hence complete. □

A metric on the space of lattices with level structure

Metric definition

4.19 Definition. We define the space $\overleftarrow{\mathcal{L}^r_N}$ as the space of all pairs (Λ, ι) of a lattice of rank $\leq r$ and an A/N-module injection $\iota : N^{-1}\Lambda/\Lambda \hookrightarrow (N^{-1}/A)^r$. Such an ι is called an *r-inverse level N structure*, or simply an inverse level N structure if the value of r is understood.

The notation above with a backwards arrow is used to remind the reader that $\overleftarrow{\mathcal{L}^r_N}$ is the collection of lattices with *inverse* level N structure.

4.20 If in particular Λ in the above definition is of rank *equal to* r, then considering the sizes of $(N^{-1}/A)^s$ and $N^{-1}\Lambda/\Lambda$, the injection ι must in fact be an isomorphism, and so the subset of $\overleftarrow{\mathcal{L}^r_N}$ with Λ of rank r is isomorphic to $\mathcal{L}^r_N$, with ι^{-1} playing the role of the relevant level N structure (hence calling ι an *inverse* level N structure).

4.21 Definition. For $(\Lambda, \iota) \in \overleftarrow{\mathcal{L}^r_N}$, we define

$$\mu_{\Lambda,\iota} : \left(N^{-1}/A\right)^r \to \mathbb{C}_\infty, \quad l \mapsto \begin{cases} e_\Lambda(\iota^{-1}(l))^{-1} & \text{if } l \in \operatorname{Im}\iota - \{0\} \\ 0 & \text{otherwise} \end{cases}.$$

4.22 Corollary. $\mu_{\Lambda,\iota}(l) \neq 0 \iff l \in \operatorname{Im}\iota - \{0\}$.

4.23 Note that in Definition 4.21, $e_\Lambda \circ \iota^{-1}$ is an A/N-module bijection from $\operatorname{Im}\iota \subseteq (N^{-1}/A)^r$ to the N-division points $\phi^\Lambda[N]$ of the Drinfeld module ϕ^Λ associated to Λ; in particular, if Λ is of rank r then $e_\Lambda \circ \iota^{-1}$ is a Drinfeld module level N structure for ϕ^Λ. So we have the following proposition:

4.24 Proposition. *We can factorise the Drinfeld module-associated polynomial ϕ^Λ_N defined in Definition 3.4 as*

$$\phi^\Lambda_N(X) = X \prod_{l \in \operatorname{Im}\iota}{}' \left(1 - \frac{X}{e_\Lambda(\iota^{-1}(l))}\right) = X \prod_{l \in (N^{-1}/A)^r} (1 - \mu_{\Lambda,\iota}(l)X).$$

Proof. $\operatorname{Im}\iota^{-1} = N^{-1}\Lambda/\Lambda$; this proves the first equality. The second follows from the definition of $\mu_{\Lambda,\iota}(l)$, which is 0 for $l \in ((N^{-1}/A)^r - \operatorname{Im}\iota) \cup \{0\}$. □

As in the case of lattices without level structure, we can define a metric on the space of lattices with level structure and find its completion. Here is our metric, related to $\mathrm{d}_{\mathcal{L}}$, and defined on the space $\overleftarrow{\mathcal{L}_N^r}$:

4.25 Definition. The metric $\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}$ on $\overleftarrow{\mathcal{L}_N^r}$ is defined by

$$\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda_1,\iota_1),(\Lambda_2,\iota_2)\big) = \mathrm{d}_{\mathcal{L}}(\Lambda_1,\Lambda_2) + \sum_{l\in(N^{-1}/A)^r} |\mu_{\Lambda_1,\iota_1}(l) - \mu_{\Lambda_2,\iota_2}(l)|.$$

4.26 Proposition. $\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}$ *is a metric.*

Proof. Symmetry and nonnegativity are easy to see. So let $(\Lambda_i,\iota_i)\in\overleftarrow{\mathcal{L}_N^r}$ for $i\in\{1,2,3\}$; then

$$\begin{aligned}
&\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda_1,\iota_1),(\Lambda_2,\iota_2)\big) + \mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda_2,\iota_2),(\Lambda_3,\iota_3)\big)\\
=&\,\mathrm{d}_{\mathcal{L}}(\Lambda_1,\Lambda_2) + \mathrm{d}_{\mathcal{L}}(\Lambda_2,\Lambda_3)\\
&+\sum_{l\in(N^{-1}/A)^r} |\mu_{\Lambda_1,\iota_1}(l) - \mu_{\Lambda_2,\iota_2}(l)| + \sum_{l\in(N^{-1}/A)^r} |\mu_{\Lambda_2,\iota_2}(l) - \mu_{\Lambda_3,\iota_3}(l)|\\
\geq&\,\mathrm{d}_{\mathcal{L}}(\Lambda_1,\Lambda_3) \quad \text{by Proposition 4.2}\\
&+\sum_{l\in(N^{-1}/A)^r} |\mu_{\Lambda_1,\iota_1}(l) - \mu_{\Lambda_2,\iota_2}(l) + \mu_{\Lambda_2,\iota_2}(l) - \mu_{\Lambda_3,\iota_3}(l)|\\
=&\,\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda_1,\iota_1),(\Lambda_3,\iota_3)\big).
\end{aligned}$$

Finally, if $\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda_1,\iota_1),(\Lambda_2,\iota_2)\big)=0$ then $\mathrm{d}_{\mathcal{L}}(\Lambda_1,\Lambda_2)=0$, so that $\Lambda_1=\Lambda_2$, and $\mu_{\Lambda_1,\iota_1}(l)=\mu_{\Lambda_2,\iota_2}(l)$ for all $l\in(N^{-1}/A)^r$. Then since $\mu_{\Lambda,\iota}(l)\neq 0 \iff l\in\operatorname{Im}\iota$, we must have that $\operatorname{Im}\iota_1=\operatorname{Im}\iota_2$. Thus for all $l\in\operatorname{Im}\iota_1-\{0\}$,

$$e_{\Lambda_1}(\iota_1^{-1}(l))^{-1} = \mu_{\Lambda_1,\iota_1}(l) = \mu_{\Lambda_1,\iota_2}(l) = e_{\Lambda_1}(\iota_2^{-1}(l))^{-1} \iff \iota_1^{-1}(l)=\iota_2^{-1}(l)$$

since $e_{\Lambda_1}:\mathbb{C}_\infty/\Lambda_1\to\mathbb{C}_\infty$ is a bijection, and so $\iota_1=\iota_2$. □

In Definition 4.25, the sets $\operatorname{Im}\iota_1$ and $\operatorname{Im}\iota_2$ may have nonempty set difference, both for $\operatorname{Im}\iota_1-\operatorname{Im}\iota_2$ and $\operatorname{Im}\iota_2-\operatorname{Im}\iota_1$; so as special cases of the above definition we have the following:

4.27 Corollary.

- *If $\Lambda_2=0$ is the zero lattice, then*

$$\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda,\iota),(0,0)\big) = \mathrm{d}_{\mathcal{L}}(\Lambda,0) + \sideset{}{'}\sum_{\lambda\in N^{-1}\Lambda/\Lambda}\left|\frac{1}{e_\Lambda(\lambda)}\right|.$$

- *If* $\operatorname{Im}\iota_2 \subseteq \operatorname{Im}\iota_1$, *then*

$$\begin{aligned}\mathrm{d}_{\overleftarrow{\mathcal{L}^r_N}}\big((\Lambda_1,\iota_1),(\Lambda_2,\iota_2)\big) &= \mathrm{d}_{\mathcal{L}}(\Lambda_1,\Lambda_2)\\ &+ \sum_{\substack{l\in\operatorname{Im}\iota_1\\ l\notin\operatorname{Im}\iota_2}} \left|e_{\Lambda_1}(\iota_1^{-1}(l))^{-1}\right| + \sideset{}{'}\sum_{l\in\operatorname{Im}\iota_2}\left|\frac{1}{e_{\Lambda_1}(\iota_1^{-1}(l))} - \frac{1}{e_{\Lambda_2}(\iota_2^{-1}(l))}\right|.\end{aligned}$$

Similarly to before, this metric space structure on $\overleftarrow{\mathcal{L}^r_N}$ induces a topology. In particular, we have the following:

4.28 Proposition. *Let* Λ *be a variable lattice of rank at most* r *with a variable* r*-inverse level* N *structure* ι. *If* $\mathrm{d}_L(\Lambda,0)\to 0$ *then* $\mathrm{d}_{\overleftarrow{\mathcal{L}^r_N}}\big((\Lambda,\iota),0\big)\to 0$.

Proof. Since $\mathrm{d}_L(\Lambda,0)\to 0$, $R=\min'_{\lambda\in\Lambda}|\lambda|$ is large. Now let $l\in\operatorname{Im}\iota-\{0\}$ and $a\in N-\{0\}$ where a is fixed. Then for a representative $\lambda'\in\iota^{-1}(l)$ we have that $e_\Lambda(\iota^{-1}(l))^{-1}=\sum_{\lambda\in\Lambda}(\lambda'+\lambda)^{-1}$, so that

$$\left|\frac{1}{a\cdot e_\Lambda(\iota^{-1}(l))}\right| \le \max_{\lambda\in\Lambda}\frac{1}{|a\lambda'+a\lambda|} \le \max_{\lambda\in\Lambda}\frac{1}{R}=\frac{1}{R},$$

since each $a\lambda'+a\lambda\in\Lambda-\{0\}$. Hence

$$\sideset{}{'}\sum_{l\in\operatorname{Im}\iota}\left|\frac{1}{e_\Lambda(\iota^{-1}(l))}\right| \le \frac{|a|(\#\operatorname{Im}\iota-1)}{R} < \frac{|a|\#(A/N)^r}{R}\to 0. \qquad \square$$

Now for rank $s\le r$, the space $\overleftarrow{\mathcal{L}^s_N}$ can be considered in multiple ways as a subspace of $\overleftarrow{\mathcal{L}^r_N}$; we now compare the metrics on this subspace both as $\overleftarrow{\mathcal{L}^s_N}$ and as a sub-metric space of $\overleftarrow{\mathcal{L}^r_N}$:

4.29 Proposition. *Any* A/N*-module injection* $\delta:(N^{-1}/A)^s\hookrightarrow(N^{-1}/A)^r$ *induces an injection* $\delta:\overleftarrow{\mathcal{L}^s_N}\hookrightarrow\overleftarrow{\mathcal{L}^r_N}$, $(\Lambda,\iota)\mapsto(\Lambda,\delta\circ\iota)$. *The metrics* $\mathrm{d}_{\overleftarrow{\mathcal{L}^s_N}}$ *and* $\mathrm{d}_{\overleftarrow{\mathcal{L}^r_N}}$ *agree on the subspace* $\overleftarrow{\mathcal{L}^s_N}$ *of* $\overleftarrow{\mathcal{L}^r_N}$.

Proof. Let $(\Lambda_1,\iota_1),(\Lambda_2,\iota_2)\in\overleftarrow{\mathcal{L}^s_N}$. Then $\delta\circ\iota_1:N^{-1}\Lambda_1/\Lambda_1\hookrightarrow(N^{-1}/A)^r$ and $\delta\circ\iota_2:N^{-1}\Lambda_2/\Lambda_2\hookrightarrow(N^{-1}/A)^r$ are A/N-module injections and thus r-inverse level N structures for Λ_1 and Λ_2 respectively, and

$$\begin{aligned}
&\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big(\delta(\Lambda_1,\iota_1),\delta(\Lambda_2,\iota_2)\big)\\
=\,&\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda_1,\delta\circ\iota_1),(\Lambda_2,\delta\circ\iota_2)\big)\\
=\,&\mathrm{d}_{\mathcal{L}}(\Lambda_1,\Lambda_2)+\sum_{l\in(N^{-1}/A)^r}|\mu_{\Lambda_1,\delta\circ\iota_1}(l)-\mu_{\Lambda_2,\delta\circ\iota_2}(l)|\\
=\,&\mathrm{d}_{\mathcal{L}}(\Lambda_1,\Lambda_2)+\sum_{l\in\operatorname{Im}\delta}|\mu_{\Lambda_1,\delta\circ\iota_1}(l)-\mu_{\Lambda_2,\delta\circ\iota_2}(l)|\\
=\,&\mathrm{d}_{\mathcal{L}}(\Lambda_1,\Lambda_2)+\sum_{l\in(N^{-1}/A)^s}|\mu_{\Lambda_1,\iota_1}(l)-\mu_{\Lambda_2,\iota_2}(l)|\\
=\,&\mathrm{d}_{\overleftarrow{\mathcal{L}_N^s}}\big((\Lambda_1,\iota_1),(\Lambda_2,\iota_2)\big).
\end{aligned}$$

□

The metric $\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}$ is thus independent of the rank r in a sense.

Metric Completeness

As in the case of level-less lattices, $\overleftarrow{\mathcal{L}_N^r}$ is the metric completion of $\mathcal{L}_N^r$ as shown by the following results:

4.30 Proposition. *$\overleftarrow{\mathcal{L}_N^r}$ is complete.*

Proof. Let $(\Lambda_n,\iota_n)_{n=0}^{\infty}$ be a Cauchy sequence in $\overleftarrow{\mathcal{L}_N^r}$. Then since

$$\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda_a,\iota_a),(\Lambda_b,\iota_b)\big)\geq \mathrm{d}_{\mathcal{L}}(\Lambda_a,\Lambda_b)$$

for any $(\Lambda_a,\iota_a),(\Lambda_b,\iota_b)\in\overleftarrow{\mathcal{L}_N^r}$, $(\Lambda_n)_{n=0}^{\infty}$ is a Cauchy sequence in $\mathcal{L}^{\leq r}$ which converges to a lattice $\Lambda\in\mathcal{L}^{\leq r}$ by Proposition 4.8. Let the Drinfeld modules ϕ^{Λ_n} correspond to the lattices Λ_n.

We now find the r-inverse level N structure ι corresponding to Λ. Since

$$\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}((\Lambda_a,\iota_a),(\Lambda_b,\iota_b))\geq\sum_{l\in(N^{-1}/A)^r}|\mu_{\Lambda_a,\iota_a}(l)-\mu_{\Lambda_b,\iota_b}(l)|$$

for any pairs $(\Lambda_a,\iota_a),(\Lambda_b,\iota_b)\in\overleftarrow{\mathcal{L}_N^r}$, we have that for each $l\in(N^{-1}/A)^r$, $(\mu_{\Lambda_n,\iota_n}(l))_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{C}_\infty$ and hence converges; we will call the limit $\mu(l)$, where $\mu:(N^{-1}/A)^r\to\mathbb{C}_\infty$. If $\mu(l)\neq 0$, then $\mu_{\Lambda_n,\iota_n}(l)\neq 0$ for large n, i.e. $l\in\operatorname{Im}\iota_n-\{0\}$ for large n. Since all the analytic functions described in this proof are additive, we have that $\mu(l_1+l_2)=\mu(l_1)+\mu(l_2)$.

Similarly to the proof of Proposition 4.8 we have that for each $a \in A$ the polynomials $\phi_a^{\Lambda_n}$ converge to ϕ_a^{Λ}. Although this convergence is initially only uniform on the unit disk, since they are polynomials it is uniform on every bounded disk. Thus although the convergence $e_{\Lambda_n} \to e_\Lambda$ is initially only uniform on the unit disk, since $e_{\Lambda_n}(az) = \phi_a^{\Lambda_n}(e_{\Lambda_n}(z))$ and $e_\Lambda(az) = \phi_a^{\Lambda}(e_\Lambda(z))$ this uniformity can be extended to any bounded disk. Thus for any nonzero $a \in A$ and l with $\mu(l) \neq 0$,

$$\begin{aligned}\phi_a^{\Lambda}(\mu(l)^{-1}) &= \left(\phi_a^{\Lambda}\left(\frac{1}{\mu(l)}\right) - \phi_a^{\Lambda_n}\left(\frac{1}{\mu(l)}\right)\right) + \left(\phi_a^{\Lambda_n}\left(\frac{1}{\mu(l)}\right) - \phi_a^{\Lambda_n}\left(\frac{1}{\mu_n(l)}\right)\right) \\ &\qquad + \phi_a^{\Lambda_n}\left(\frac{1}{\mu_n(l)}\right) \\ &= o(1) + o(1) + \phi_a^{\Lambda_n}(e_{\Lambda_n}(\iota_n(l))) \quad \text{as } n \to \infty \\ &= e_{\Lambda_n}(a\iota_n(l)) + o(1) = e_{\Lambda_n}(\iota_n(al)) + o(1) = \mu_n(al)^{-1} + o(1) \\ &\to \mu(al)^{-1}.\end{aligned}$$

Since this is true for any $a \in A$, each nonzero $\mu(l)^{-1} \in \phi^\Lambda[N]$, and also $1/\mu$ has the desired A/N-module structure. We may then define the lattice r-inverse level N structure $\iota = e_\Lambda^{-1} \circ (1/\mu)$, with $(\Lambda_n, \iota_n) \to (\Lambda, \iota)$ as desired. □

4.31 Proposition. *$\mathcal{L}_N^r$ is a dense subset of $\overleftarrow{\mathcal{L}_N^r}$.*

Proof. Let $(\Lambda, \iota) \in \overleftarrow{\mathcal{L}_N^r}$ be of rank s with $0 \leq s \leq r$, let $f \in F$ with $|f| > 1$, and let $\omega_{s+1}, \dots, \omega_r \in \mathbb{C}_\infty$ be F_∞-linearly independent with Λ, i.e. the F_∞-vector space $F_\infty \otimes \Lambda + F_\infty\omega_{s+1} + \cdots + F_\infty\omega_r$ has full dimension r. Also, let $\Lambda' = A\omega_{s+1} + \cdots + A\omega_r$ be a lattice of rank $r-s$ and let $\Lambda_n = \Lambda + f^n\Lambda'$ be lattices of rank r for $n \in \mathbb{N}$. To define the corresponding level structures, note that by Paragraph 2.6 and the proof of Proposition 4.44 there is an A/N-module isomorphism

$$j : \left(N^{-1}/A\right)^r \hookrightarrow\!\!\!\!\rightarrow \left(N^{-1}/A\right)^r \quad \text{with} \quad \mathrm{Im}\left(j^{-1} \circ \iota\right) = (N^{-1}/A)^s \oplus \{0\}^{r-s}.$$

Also, $N^{-1}\Lambda_n/\Lambda_n \simeq N^{-1}\Lambda/\Lambda \oplus f^n(N^{-1}\Lambda'/\Lambda')$. We then define the inverse level N structures ι' for Λ' and ι_n for Λ_n by

$$\begin{aligned}\iota'(l_{s+1}\omega_{s+1} + \cdots + l_r\omega_r) &= (0, \dots, 0, l_{s+1}, \dots, l_r) \quad \text{for } l_i \in N^{-1}/A, \\ (j^{-1} \circ \iota_n)(\lambda + f^n\lambda') &= (j^{-1} \circ \iota)(\lambda) + \iota'(\lambda')\end{aligned}$$

for $\lambda \in N^{-1}\Lambda/\Lambda, \lambda' \in N^{-1}\Lambda'/\Lambda'$. Note that since each Λ_n has full rank, each ι_n is a bijection. We will now show that $(\Lambda_n, \iota_n) \to (\Lambda, \iota)$.

Firstly, as in the proof of Proposition 4.10 we have that $\Lambda_n \to \Lambda$; in particular, $\sup_{|z|\le 1}|e_{\Lambda_n}(z) - e_\Lambda(z)| \to 0$ and $R_n = \min'_{\lambda\in e_\Lambda(\Lambda_n)}|\lambda| \to \infty$. In fact, since the Drinfeld modules $\phi_a^{\Lambda_n}$ converge to ϕ_a^Λ for all $a \in A$ we have that

$$\sup_{|z|\le M} |e_{\Lambda_n}(z) - e_\Lambda(z)| \to 0$$

for any $M > 0$, and by a similar argument to that for $R_n \to \infty$ we have that $R'_n = \min'_{\lambda\in e_\Lambda(N^{-1}\Lambda_n)} |\lambda| \to \infty$. Next, let $l \in (N^{-1}/A)^r$. Then for $l \in \operatorname{Im}\iota - \{0\}$, $\iota_n^{-1}(l) = \iota^{-1}(l)$ and so

$$\mu_{\Lambda_n,\iota_n}(l) - \mu_{\Lambda,\iota}(l) = \frac{1}{e_{\Lambda_n}(\iota^{-1}(l))} - \frac{1}{e_\Lambda(\iota^{-1}(l))}.$$

Now $\iota^{-1}(l) \in N^{-1}\Lambda/\Lambda - \{0\}$, so fixing a representative we have that

$$e_{\Lambda_n}(\iota^{-1}(l)) \to e_\Lambda(\iota^{-1}(l)) \neq 0; \quad \text{hence} \quad \mu_{\Lambda_n,\iota_n}(l) - \mu_{\Lambda,\iota}(l) \to 0.$$

Finally, for $l \in \operatorname{Im}\iota_n - \operatorname{Im}\iota$, note that $\iota_n^{-1}(l) = \lambda + f^n\lambda'$ for $\lambda \in N^{-1}\Lambda/\Lambda$ and $\lambda' \in N^{-1}\Lambda'/\Lambda' - \{0\}$. Using λ_0 and λ'_0 for representatives of these classes,

$$\begin{aligned}\mu_{\Lambda_n,\iota_n}(l) &= \frac{1}{e_{\Lambda_n}(\iota_n^{-1}(l))} = \frac{1}{e_{\Lambda_n}(\lambda + f^n\lambda')}\\ &= \sum_{\lambda_n\in\Lambda_n} \frac{1}{\lambda_n + \lambda_0 + f^n\lambda'_0} = \sum_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda'} \frac{1}{(\lambda+\lambda_0) + f^n(\lambda'+\lambda'_0)};\end{aligned}$$

we show that these fractions go to zero uniformly. Indeed, in the above $\lambda' + \lambda'_0 \in N^{-1}\Lambda' - \{0\}$ and $\lambda + \lambda_0 \in N^{-1}\Lambda$. So for any fixed nonzero $a \in N$,

$$\begin{aligned}|(\lambda+\lambda_0) + f^n(\lambda'+\lambda'_0)| &\ge \mathrm{d}\big(f^nN^{-1}\Lambda' - \{0\}, N^{-1}\Lambda\big)\\ &\ge \mathrm{d}\big(f^na^{-1}\Lambda' - \{0\}, F_\infty\Lambda\big)\\ &= |f|^n|a|^{-1}\,\mathrm{d}(\Lambda' - \{0\}, \Lambda)\\ &=: R\cdot|f|^n,\end{aligned}$$

where $R > 0$ since the ω_i are F_∞-linearly independent with Λ. Hence $|\mu_{\Lambda_n,\iota_n}(l)| \le R^{-1}|f|^{-n} \to 0$, as desired. □

4.32 Theorem. *The space* $\overleftarrow{\mathcal{L}_N^r}$ *is the completion of* $\mathcal{L}_N^r$.

Proof. By Propositions 4.30 and 4.31, $\mathcal{L}_N^r$ is dense in the complete $\overleftarrow{\mathcal{L}_N^r}$. □

Topology of the irreducible components

So we see that the space $\mathcal{L}_N^r$ has boundary $\partial\,\mathcal{L}_N^r = \partial\,\overleftarrow{\mathcal{L}_N^r}$ in the complete metric space $\overleftarrow{\mathcal{L}_N^r}$ consisting of the (Λ, ι) with Λ of rank strictly less than r. Denoting this boundary by ∂_N^r, we will see that each of the irreducible components of $\mathcal{L}_N^r$ possesses the same boundary.

First, let us look closer at ∂_N^r:

4.33 Proposition. *∂_N^r is closed as a subset of $\overleftarrow{\mathcal{L}_N^r}$, so that $\mathcal{L}_N^r = \overleftarrow{\mathcal{L}_N^r} - \partial_N^r$ is open.*

Proof. Let $(\Lambda_n, \iota_n)_{n=0}^{\infty}$ be a Cauchy sequence in ∂_N^r, so that each Λ_n has rank strictly less than r. Then since $\overleftarrow{\mathcal{L}_N^r}$ is complete, this sequence has a limit $(\Lambda, \iota) \in \overleftarrow{\mathcal{L}_N^r}$. Now by the definitions of $\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}$ and $\mathrm{d}_{\mathcal{L}}$, we also have that $\Lambda_n \to \Lambda$; hence by Corollary 4.9 we have that Λ has rank strictly less than r, so that $(\Lambda, \iota) \in \partial_N^r$ as desired. □

For the rest of the results in this subsection, we let C denote a fixed irreducible component of $\mathcal{L}_N^r$.

4.34 Proposition. *C is a dense subset of $C \cup \partial_N^r$.*

Proof. Let $(\Lambda, \iota) \in \partial_N^r$ with $\Lambda = I_1\psi_1 + \cdots + I_s\psi_s$ of rank $s < r$, let $f \in F$ with $|f| > 1$, and let $\psi_{s+1}, \ldots, \psi_r \in \mathbb{C}_\infty$ be F_∞-linearly independent with Λ. Also let $I \in \mathcal{J}(A)$ such that $[I_1 \cdots I_s I]_{\mathrm{Cl}(F)} = \pi_N(C)$, let

$$\Lambda' = A\psi_{s+1} + \cdots + A\psi_{r-1} + I\psi_r$$

be a lattice of rank $r - s$ and let $\Lambda_n = \Lambda + f^n\Lambda'$ be lattices of rank r for $n \in \mathbb{N}$. Then each Λ_n has

$$\pi(\Lambda_n) = [I_1 \cdots I_s A \cdots A I]_{\mathrm{Cl}(F)} = \pi_N(C),$$

so that by Propositions 3.30 and 3.32, for any level structure ι_0 for Λ_n there is an $x \in (A/N)^\times/\mathbb{F}_q^\times$ such that $i_N(x)\det(\Lambda_n, \iota_0^{-1}) = \det(C)$; here $\det(\Lambda_n, \iota_0^{-1})$ denotes $[\det g] \in F^\times \Big\backslash \left(\mathbb{A}_F^{fin}\right)^\times \Big/ \det K(N)$ for $\Theta([(\psi, g)]) = (\Lambda, \iota_0^{-1})$, and $\det(C)$ denotes the common value of $\det(\Lambda'', \alpha'')$ for $(\Lambda'', \alpha'') \in C$.

Now to define the corresponding level structures for Λ_n; let $x_n \in (A/N)^\times$ be chosen later for each $n \in \mathbb{N}$. Note that by Paragraph 2.6 and the proof of Proposition 4.44 there is an A/N-module isomorphism

$$j : \left(N^{-1}/A\right)^r \hookrightarrow\!\!\!\!\rightarrow \left(N^{-1}/A\right)^r \quad \text{with} \quad \mathrm{Im}\left(j^{-1} \circ \iota\right) = (N^{-1}/A)^s \oplus \{0\}^{r-s}.$$

Also, $N^{-1}\Lambda_n/\Lambda_n \simeq N^{-1}\Lambda/\Lambda \oplus f^n(N^{-1}\Lambda'/\Lambda')$. We then define the inverse level N structures ι'_n for Λ' and ι_n for Λ_n by

$$\begin{aligned} \iota'_n(l_{s+1}\omega_{s+1} + \cdots + l_r\omega_r) &= (0, \ldots, 0, l_{s+1}, \ldots, x_n l_r) \quad \text{for } l_i \in N^{-1}/A, \\ (j^{-1} \circ \iota_n)(\lambda + f^n\lambda') &= (j^{-1} \circ \iota)(\lambda) + \iota'_n(\lambda') \end{aligned}$$

for $\lambda \in N^{-1}\Lambda/\Lambda$ and $\lambda' \in N^{-1}\Lambda'/\Lambda'$, noting the appearance of x_n. Note that since each Λ_n has full rank, each ι_n is a bijection, and by varying x_n for each n we can ensure that $(\Lambda_n, \iota_n) \in C$ for each $n \in \mathbb{N}$.

The rest of the proof proceeds as in the proof of Proposition 4.31, showing that $(\Lambda_n, \iota_n) \to (\Lambda, \iota)$ as $n \to \infty$. Hence each point in ∂^r_N is the limit of a sequence in C, as desired. □

4.35 Proposition. *C is open in $\overleftarrow{\mathcal{L}^r_N}$.*

Proof. This proof will build on the proof of Proposition 4.16. Let $(\Lambda, \iota) \in C$ be given, and let $M > 1$ be such that every $\overline{\lambda} \in N^{-1}\Lambda/\Lambda$ has a representative $\lambda \in N^{-1}\Lambda$ with $|\lambda| < M$, the existence of such an M guaranteed by Lemma 4.6. We also temporarily define $|N^{-1}\Lambda| = \min'_{\lambda \in N^{-1}\Lambda}|\lambda|$, fix a large enough $R > 0$ that in addition to the conditions in the other proof the ball $\mathrm{B}_R = \mathrm{B}(0, R)$ contains a complete set of such representatives of $N^{-1}\Lambda/\Lambda$. Finally we choose an $\epsilon > 0$ such that

$$\epsilon^{1/3} < \min\left\{1, \frac{|N^{-1}\Lambda|}{M}, \min'_{l \in (N^{-1}/A)^r} \frac{1}{|e_\Lambda(\iota^{-1}(l))|}\right\},$$

then consider a $(\Lambda', \iota') \in \overleftarrow{\mathcal{L}^r_N}$ such that

$$\sup_{|z| \le R} |e_{\Lambda'}(z) - e_\Lambda(z)| + \sum_{l \in (N^{-1}/A)^r}' \left|\mu_{\Lambda',\iota'}(l) - e_\Lambda(\iota^{-1}(l))^{-1}\right| < \epsilon,$$

so that

4.36 $$\sup_{|z| \le R} |e_{\Lambda'}(z) - e_\Lambda(z)| < \epsilon$$

and

$$\left|\mu_{\Lambda',\iota'}(l) - e_\Lambda(\iota^{-1}(l))^{-1}\right| < \epsilon \quad \text{for each } l \in \left(N^{-1}/A\right)^r,$$

and show that $(\Lambda', \iota') \in C$.

As in the proof of Proposition 4.16, we have an A-module isomorphism $\overline{f} : \Lambda \hookrightarrow\!\!\!\!\rightarrow \Lambda'$ such that $\left|\lambda - \overline{f}(\lambda)\right| < \epsilon$ for each $\lambda \in \Lambda \cap \mathrm{B}_R$; this can be naturally extended to an A-module bijection $N^{-1}\Lambda \hookrightarrow\!\!\!\!\rightarrow N^{-1}/\Lambda'$ with the same property; we will again use the notation $\overline{f}$ for this extended bijection.

Since Λ' is then also of full rank r, $\operatorname{Im}\iota' = (N^{-1}/A)^r$ and so the $\mu_{\Lambda',\iota'}(l)$ are nonzero and equal to $e_{\Lambda'}(\iota'^{-1}(l))^{-1}$ for $l \neq 0$; we can thus define the level N structures $\alpha = \iota^{-1}$ and $\alpha' = \iota'^{-1}$ for convenience. $\overline{f}$ also defines a bijection $N^{-1}\Lambda/\Lambda \hookrightarrow\!\!\!\!\rightarrow N^{-1}\Lambda'/\Lambda'$, for which we will use the same symbol $\overline{f}$; to show that (Λ', α') lies in the same component as (Λ, α) it suffices to show that $\alpha' = \overline{f} \circ \alpha$, since then $(\Lambda, \alpha) = \Theta([\psi, g]) \implies (\Lambda', \alpha') = \Theta([\psi', g])$ where $|\psi - \psi'|$ is small.

Now by the choice of ϵ we have that $e_\Lambda(\alpha(l))^{-1} > \epsilon^{1/3}$ for each $l \in (N^{1-}/A)^r$, so that the corresponding $e_{\Lambda'}(\alpha'(l))^{-1} > \epsilon^{1/3}$ too. Thus we have that

$$\begin{aligned} &\left|\frac{1}{e_\Lambda(\alpha(l))} - \frac{1}{e_{\Lambda'}(\alpha'(l))}\right| < \epsilon \\ \implies & |e_\Lambda(\alpha(l)) - e_{\Lambda'}(\alpha'(l))| = \left|\frac{1}{e_\Lambda(\alpha(l))} - \frac{1}{e_{\Lambda'}(\alpha'(l))}\right| \cdot |e_\Lambda(\alpha(l))| \cdot |e_{\Lambda'}(\alpha'(l))| \\ & \qquad < \epsilon \cdot \frac{1}{\epsilon^{1/3}} \cdot \frac{1}{\epsilon^{1/3}} = \epsilon^{1/3}. \end{aligned}$$

Letting $\lambda \in N^{-1}\Lambda \cap \mathrm{B}_R$ and $\lambda' \in N^{-1}\Lambda' \cap \mathrm{B}_R$ be representatives for $\alpha(l) \in N^{-1}\Lambda/\Lambda$ and $\alpha'(l) \in N^{-1}\Lambda'/\Lambda'$ respectively, we have by Equation 4.36 that $|e_{\Lambda'}(\lambda') - e_\Lambda(\lambda')| < \epsilon$, so that

$$\begin{aligned} |e_\Lambda(\lambda - \lambda')| &= |e_\Lambda(\lambda) - e_\Lambda(\lambda')| \\ &\leq \max\{|e_\Lambda(\lambda) - e_{\Lambda'}(\lambda')|, |e_{\Lambda'}(\lambda') - e_\Lambda(\lambda')|\} \\ &< \max\{\epsilon^{1/3}, \epsilon\} = \epsilon^{1/3}. \end{aligned}$$

Similarly to the proof of Proposition 4.16, from the bound $|N^{-1}\Lambda| > \epsilon^{1/3}$ we have that $|e_\Lambda(z)| < \epsilon^{1/3} \implies z \in \mathrm{B}(\lambda_0, \epsilon^{1/3})$ for some $\lambda_0 \in \Lambda$; thus $|\lambda - \lambda' - \lambda_0| < \epsilon^{1/3}$ for some $\lambda_0 \in \Lambda \cap \mathrm{B}_R$. Replacing λ by $\lambda + \lambda_0$, we still have $\lambda \in \alpha(l) \cap \mathrm{B}_R$, and now $|\lambda - \lambda'| < \epsilon$. Now we have that $\left|\overline{f}(\lambda) - \lambda\right| < \epsilon^{1/3}$, so now $\left|\overline{f}(\lambda) - \lambda'\right| < \epsilon^{1/3}$; but $\overline{f}(\lambda) - \lambda' \in N^{-1}\Lambda'$, so from the bound $|N^{-1}\Lambda'| = |N^{-1}\Lambda| > \epsilon^{1/3}$ we get that $\overline{f}(\lambda) = \lambda' \implies \overline{f}(\alpha(l)) = \alpha'(l)$.

Since this is true for all $l \in (N^{-1}/A)^r$, we have that $\alpha' = \overline{f} \circ \alpha$ as desired. □

4.37 Corollary. *$C \cup \partial^r_N$ is closed in $\overleftarrow{\mathcal{L}^r_N}$.*

Proof. If $\mathcal{D}$ denotes the set of irreducible components of $\mathcal{L}^r_N$, then

$$\overleftarrow{\mathcal{L}^r_N} - (C \cup \partial^r_N) = \mathcal{L}^r_N - C = \bigcup_{C' \in \mathcal{D} - \{C\}} C' \quad \text{is open.}$$ □

4.38 Theorem. *$C \cup \partial^r_N$ is the completion of C, i.e. the closure of C in $\overleftarrow{\mathcal{L}^r_N}$.*

Proof. By Proposition 4.34, C is dense in $C \cup \partial_N^r$, which is closed in $\overleftarrow{\mathcal{L}_N^r}$ by Corollary 4.37 and hence complete. $\square$

Boundary strata for lattices with level structure

We now decompose $\overleftarrow{\mathcal{L}_N^r}$ into a disjoint union of $\mathcal{L}_N^s$ for $0 \le s \le r$, but first we will need some ring- and module-theoretic results:

The structure of finitely generated free A/N-modules

4.39 Definition. Let S be a subset of an R-module M, and let R_{fin}^S denote the R-module of sequences $(c_s)_{s \in S}$ in R indexed by S where all but finitely many of the c_s are zero. There is an R-module homomorphism

$$R_{fin}^S \to M, \quad (c_s)_{s \in S} \mapsto \sum_{s \in S} c_s s.$$

S is said to be *linearly independent* if the above map is an injection, i.e.

$$\sum_{s \in S} c_s s = 0 \implies \text{ each } c_s = 0 \quad \text{for } (c_s)_{s \in S} \in R_{fin}^S,$$

is said to *span* M if the above map is a surjection, i.e.

$$\sum_{s \in S} Rs = M,$$

and is said to be a *basis* if it is both linearly independent and spans M, so that the above map is an isomorphism. M is said to be *free* if it has a basis.

4.40 Lemma. *If S is a linearly independent subset of the free A/N-module $(A/N)^r$, then S can be extended to a basis.*

Proof. This proof is largely due to [Wof|SE]. Since A is a Dedekind domain, N can be factorised into a product of prime ideals as $N = \prod_i \mathfrak{p}_i^{d_i}$, and hence by the Chinese Remainder Theorem $A/N \simeq \prod_i A/\mathfrak{p}_i^{d_i}$. So the image of S under each projection $\pi_i : (A/N)^r \twoheadrightarrow \left(A/\mathfrak{p}_i^{d_i}\right)^r$ is also linearly independent in $\left(A/\mathfrak{p}_i^{d_i}\right)^r$; indeed, suppose that $\sum_{s \in S} c_{s,i}\pi_i(s) = 0$ in $\left(A/\mathfrak{p}_i^{d_i}\right)^r$for $c_{s,i} \in A/\mathfrak{p}_i^{d_i}$, or equivalently for lifts $c_s \in A/N$ of the $c_{s,i}$ that $\sum_{s \in S} c_s s \in \mathfrak{p}_i^{d_i}(A/N)^r$. Then by the Chinese Remainder Theorem there is an $m \in A/N$ such that $\pi_i(m) = 1$ and $m \in (N/\mathfrak{p}_i^{d_i})A/N$, or equivalently such that $m \equiv_{\mathfrak{p}_i^{d_i}} 1$ and $m \equiv_{\mathfrak{p}_j^{d_j}} 0$ for

$j \neq i$. Then $\sum_{s \in S} mc_s s = 0$, so that since S is linearly independent we have that each $mc_s \equiv_N 0$. In particular, modulo $\mathfrak{p}_i^{d_i}$ we have that each $c_s \equiv_{\mathfrak{p}_i^{d_i}} mc_s \equiv_{\mathfrak{p}_i^{d_i}} 0$.

Now if we can extend the image of S to a basis for each $\left(A/\mathfrak{p}_i^{d_i}\right)^r$, we will have a basis for $(A/N)^r$. So without loss of generality consider the case where $N = \mathfrak{p}^d$, and let S' be the image of S under the projection $(A/\mathfrak{p}^d)^r \twoheadrightarrow (A/\mathfrak{p})^r$, the codomain being a finite-dimensional vector space over the field $A/\mathfrak{p}$. Then S' is also linearly independent over $A/\mathfrak{p}$; indeed, suppose that we have scalars $c_s \in A$ such that $\sum_{s \in S} c_s s \in \mathfrak{p}$. Then for each $p \in \mathfrak{p}$, $\sum_{s \in S} p^{d-1} c_s s \in \mathfrak{p}^d$, whence each $p^{d-1} c_s \in \mathfrak{p}^d$ since S is linearly independent over $A/\mathfrak{p}^d$. Since this is true for all $p \in \mathfrak{p}$, we have that each $c_s \in \mathfrak{p}$, and so S' is linearly independent over $A/\mathfrak{p}$ and can hence be extended to a basis for $(A/\mathfrak{p})^r$. Taking a preimage under the projection $(A/\mathfrak{p}^d)^r \twoheadrightarrow (A/\mathfrak{p})^r$, we get an extension T of S of cardinality r such that T is $A/\mathfrak{p}^d$-linearly independent.

This T must be a basis. Indeed, since T is linearly independent over $A/\mathfrak{p}^d$, $\#(\sum_{t \in T} (A/\mathfrak{p}^d) t) = \#(A/\mathfrak{p}^d)^{\#T} = \#\left((A/\mathfrak{p}^d)^r\right)$, and so since r and $A/\mathfrak{p}^d$ are finite, T must be a generating set. This completes the proof. $\square$

4.41 Proposition.

1. *If $f : (A/N)^r \twoheadrightarrow (A/N)^s$ is an A/N-module surjection, then the kernel $\ker f \simeq (A/N)^{r-s}$ is free.*

2. *If $i : (A/N)^s \hookrightarrow (A/N)^r$ is an A/N-module injection, then the quotient $(A/N)^r / \operatorname{Im} i \simeq (A/N)^{r-s}$ is free.*

Proof.

1. Since $(A/N)^s$ is free it is projective and hence f has a right inverse $g : (A/N)^s \hookrightarrow (A/N)^r$. Then for a basis S of $(A/N)^s$, $g(S)$ is linearly independent in $(A/N)^r$, and so by Lemma 4.40 can be extended to a basis $T = \{t_1, \ldots, t_r\}$, where without loss of generality $S = \{f(t_1), \ldots, f(t_s)\}$ and $f(t_i) = 0$ for $s < i \leq r$. Then $\ker f \simeq \sum_{s < i \leq r} (A/N) t_i \simeq (A/N)^{r-s}$.

2. Let B be a basis for $(A/N)^s$. Then $i(B)$ is linearly independent in $(A/N)^r$ and so by Lemma 4.40 can be extended to a basis $T = \{t_1, \ldots, t_r\}$ where without loss of generality $i(B) = \{t_1, \ldots, t_s\}$. Then if h is the projection $h : (A/N)^r \twoheadrightarrow (A/N)^r / \operatorname{Im} i$, the quotient has as a basis $\{h(t_{s+1}), \ldots, h(t_r)\}$ and thus is free. $\square$

4.42 Definition. For each integer r, s with $0 \leq s \leq r$, we define the sets $\operatorname{Sur}_N^{r,s} =$

$\{(A/N)^r \twoheadrightarrow (A/N)^s\}$ of A/N-module surjections (epimorphisms) and $\mathrm{Inj}_N^{s,r} = \{(A/N)^s \hookrightarrow (A/N)^r\}$ of A/N-module injections (monomorphisms).

4.43 By Proposition 4.41, the elements of $\mathrm{Sur}_N^{r,s}$ and $\mathrm{Inj}_N^{s,r}$ are actually all split epimorphisms and split monomorphisms, respectively.

4.44 Proposition.

1. *The natural left action of* $\mathrm{GL}_s(A/N)$ *on* $\mathrm{Sur}_N^{r,s}$ *is free.*
2. *The natural right action of* $\mathrm{GL}_r(A/N)$ *on* $\mathrm{Sur}_N^{r,s}$ *is transitive.*
3. *The natural right action of* $\mathrm{GL}_s(A/N)$ *on* $\mathrm{Inj}_N^{s,r}$ *is free.*
4. *The natural left action of* $\mathrm{GL}_r(A/N)$ *on* $\mathrm{Inj}_N^{s,r}$ *is transitive.*

Proof.

1. Let $f \in \mathrm{Sur}_N^{r,s}$ and $\gamma \in \mathrm{GL}_s(A/N)$ such that $\gamma \circ f = f$. Then since f is split it has a right inverse, yielding that $\gamma = 1$ is the identity map.
2. Consider two surjections $f_1, f_2 \in \mathrm{Sur}_N^{r,s}$ with right inverses $g_1, g_2 \in \mathrm{Inj}_N^{s,r}$ which exist by Paragraph 4.43. We also consider the associated kernel injections $i_1, i_2 \in \mathrm{Inj}_N^{r-s,r}$ with left inverses $h_1, h_2 \in \mathrm{Sur}_N^{r,r-s}$ which make the two split exact sequences $(A/N)^{r-s} \xhookrightarrow{i_{1,2}} (A/N)^r \xrightarrow{f_{1,2}} (A/N)^s$, with g_1, g_2 and h_1, h_2 included, into biproduct diagrams (the existence of these also guaranteed by Paragraph 4.43 and the proof of Proposition 4.41).

4.45
$$\begin{tikzcd}
& (A/N)^r \arrow[dl, two heads, shift right, "h_1"'] \arrow[dr, two heads, shift left, "f_1"] \arrow[dd, hook, dashed, shift left, "j_2"] & \\
(A/N)^{r-s} \arrow[ur, hook, shift right, "i_1"'] \arrow[dr, hook, shift left, "i_2"] & & (A/N)^s \arrow[ul, hook, shift left, "g_1"] \arrow[dl, hook, shift right, "g_2"'] \\
& (A/N)^r \arrow[ul, two heads, shift left, "h_2"] \arrow[ur, two heads, shift right, "f_2"'] \arrow[uu, hook, dashed, shift left, "j_1"] &
\end{tikzcd}$$

Then we define the maps $j_1, j_2 : (A/N)^r \to (A/N)^r$ by $j_1 = g_1 \circ f_2 + i_1 \circ h_2$ and $j_2 = g_2 \circ f_1 + i_2 \circ h_1$, which we will show to be elements of $\mathrm{GL}_r(A/N)$ such that $f_2 = f_1 \circ j_1$ and $f_1 = f_2 \circ j_2$, finishing the proof. Firstly,

$$f_1 \circ j_1 = f_1 \circ g_1 \circ f_2 + f_1 \circ i_1 \circ h_2 = 1 \circ f_2 + 0 \circ h_2 = f_2,$$

and similarly $f_1 = f_2 \circ j_2$. Finally, to show that they are invertible, we show that they are mutually inverse:

$$\begin{aligned} j_1 \circ j_2 &= g_1 \circ (f_2 \circ g_2) \circ f_1 + g_1 \circ (f_2 \circ i_2) \circ h_1 \\ &\quad + i_1 \circ (h_2 \circ g_2) \circ f_1 + i_1 \circ (h_2 \circ i_2) \circ h_1 \\ &= g_1 \circ 1 \circ f_1 + g_1 \circ 0 \circ h_1 + i_1 \circ 0 \circ f_1 + i_1 \circ 1 \circ h_1 \\ &= g_1 \circ f_1 + i_1 \circ h_1 \\ &= 1 \end{aligned}$$

since f_1, g_1, h_1, and i_1 form a biproduct diagram; that $j_2 \circ j_1 = 1$ is shown similarly.

3. Let $i \in \mathrm{Inj}_N^{s,r}$ and $\gamma \in \mathrm{GL}_s(A/N)$ such that $i \circ \gamma = i$. Then since i is split it has a left inverse, yielding that $\gamma = 1$.

4. Proven similarly to the second item above. □

We now turn to counting the cardinalities of the sets $\mathrm{Sur}_N^{r,s}$ and $\mathrm{Inj}_N^{s,r}$:

4.46 Proposition.

$$\#\mathrm{Inj}_N^{s,r} = \#\mathrm{Sur}_N^{r,s} = \#\mathrm{GL}_r(A/N) \Big/ \#\mathrm{GL}_{r-s}(A/N) \cdot |N|^{s(r-s)}$$

Proof. By Proposition 4.44, the right action of $G = \mathrm{GL}_r(A/N)$ on $\mathrm{Sur}_N^{r,s}$ is transitive, and so the cardinality of $\mathrm{Sur}_N^{r,s}$ is the cardinality of G divided by the size of any stabiliser group; so we calculate the size of the stabiliser group G_f of the surjection $f(x_1, \dots, x_r) = (x_1, \dots, x_s)$.

So let $\gamma \in G_f$ such that $f \circ \gamma = f$, and consider γ as an $r \times r$ matrix with elements in A/N, using the canonical basis $(e_i)_{i=1}^r$ on $(A/N)^r$. Considering f as an $s \times r$ matrix $\left(\begin{smallmatrix} I_s & 0 \end{smallmatrix} \right)$, consisting of an $s \times s$ identity matrix and an $s \times (r-s)$ zero matrix, the condition $f \circ \gamma = f$ is equivalent to the first s rows of γ being the same as f. So we can write γ as $\left(\begin{smallmatrix} I_s & 0 \\ A & B \end{smallmatrix} \right)$, where A and B are $(r-s) \times s$ and $(r-s) \times (r-s)$ matrices, respectively.

Now the only remaining condition is that γ be an automorphism, i.e. that $\det \gamma \in (A/N)^\times \iff \det B \in (A/N)^\times \iff B \in \mathrm{GL}_{r-s}(A/N)$, with no condition on A (which then has $\#(A/N)^{s \times (r-s)} = |N|^{s(r-s)}$ possibilities). So finally, the first cardinality in question is equal to

$$\frac{\#\mathrm{GL}_r(A/N)}{\#\mathrm{GL}_r(A/N)_f} = \frac{\#\mathrm{GL}_r(A/N)}{\#\mathrm{GL}_{r-s}(A/N)|N|^{s(r-s)}}.$$

Similarly, the left action of G on $\mathrm{Inj}_N^{s,r}$ is transitive, so we calculate the size of the stabiliser group G_i of the injection $i(x_1, \dots, x_s) = (x_1, \dots, x_s, 0, \dots, 0)$.

Considering a $\gamma \in G_i$ such that $\gamma \circ i = i$ as an $r \times r$ matrix with elements in A/N, with i as an $r \times s$ consisting of an $s \times s$ matrix and an $(r-s) \times s$ zero matrix, by transposing this equation we see that the number of allowed γ is equal to the previous case. Hence $\#\mathrm{Inj}_N^{s,r} = \#\mathrm{Sur}_N^{r,s}$. □

4.47 Corollary. *If Λ is a lattice of rank $s \leq r$, then the number of r-inverse level N structures for Λ is equal to*

$$\#\mathrm{GL}_r(A/N) \Big/ \#\mathrm{GL}_{r-s}(A/N) \cdot |N|^{s(r-s)}.$$

Proof. We want to count the A/N-module injections $N^{-1}\Lambda/\Lambda \hookrightarrow (N^{-1}/A)^r$. By Paragraph 2.6, this is equal to the number of A/N-module injections $(A/N)^s \hookrightarrow (A/N)^r$; i.e. the cardinality of $\mathrm{Inj}_N^{s,r}$. □

4.48 Definition. For an ideal N of A and integers r, s with $0 \leq s \leq r$, we let $\mathrm{Free}_N^{s,r}$ be the set of free A/N-submodules of $(A/N)^r$ of rank s, and define $\mathrm{Free}_N^r = \sqcup_{s=0}^r \mathrm{Free}_N^{s,r}$.

Also, we say that

$$\delta : \mathrm{Free}_N^r \to \sqcup_{s=0}^r \mathrm{Inj}_N^{s,r},\ U \mapsto \delta_U$$

is an *injective selection of* Free_N^r if for each $U \in \mathrm{Free}_N^{s,r}$, $\delta_U \in \mathrm{Inj}_N^{s,r}$ has $\mathrm{Im}\,\delta_U = U$. Similarly,

$$\epsilon : \mathrm{Free}_N^r \to \sqcup_{s=0}^r \mathrm{Sur}_N^{r,s},\ U \mapsto \epsilon_U$$

is a *surjective selection of* Free_N^r if for each $U \in \mathrm{Free}_N^{s,r}$, $\epsilon_U \in \mathrm{Sur}_N^{r,s}$ has $\ker \epsilon_U = U$.

4.49 Proposition. *There are bijections*

$$\begin{aligned} \mathrm{Inj}_N^{s,r}/\mathrm{GL}_s(A/N) &\hookrightarrow\!\!\!\rightarrow \mathrm{Free}_N^{s,r},\quad [i] \mapsto \mathrm{Im}\, i \quad \textit{and} \\ \mathrm{GL}_s(A/N)\backslash \mathrm{Sur}_N^{r,s} &\hookrightarrow\!\!\!\rightarrow \mathrm{Free}_N^{s,r},\quad [f] \mapsto \ker f. \end{aligned}$$

Proof. We will prove the first bijection; the second is proven analogously. By Proposition 4.41 $\mathrm{Im}\, i \in \mathrm{Free}_N^{s,r}$ for each $i \in \mathrm{Inj}_N^{s,r}$, and if two $i_1, i_2 \in \mathrm{Inj}_N^{s,r}$ satisfy $I = \mathrm{Im}\, i_1 = \mathrm{Im}\, i_2$, then they induce bijections $i_1', i_2' : (A/N)^s \hookrightarrow\!\!\!\rightarrow I$; thence $\gamma = i_1'^{-1} \circ i_2' \in \mathrm{GL}_s(A/N)$ satisfies $i_1 \circ \gamma = i_2$ and so i_1, i_2 are equivalent under the action of $\mathrm{GL}_s(A/N)$. Moreover, $\mathrm{Im}\, i = \mathrm{Im}(i \circ \gamma)$ for $\gamma \in \mathrm{GL}_s(A/N)$ and $i \in \mathrm{Inj}_N^{s,r}$. Also, for any $U \in \mathrm{Free}_N^{s,r}$, U has a basis by Lemma 4.40; thus $(A/N)^s \simeq U \subseteq (A/N)^r$, and the composition $i \in \mathrm{Inj}_N^{s,r}$ of these relations has $\mathrm{Im}\, i = U$. This proves the bijection. □

4.50 By Proposition 4.49, injective and surjective selections of Free_N^r exist, by choosing an element in each class of $\mathrm{Inj}_N^{s,r}/\mathrm{GL}_s(A/N)$ and $\mathrm{GL}_s(A/N)\backslash\mathrm{Sur}_N^{r,s}$ for s between 0 and r, respectively.

Decomposition of $\overleftarrow{\mathcal{L}_N^r}$ into a union of $\mathcal{L}_N^s$

4.51 For the following theorem, we fix isomorphisms $(N^{-1}/A)^s \hookrightarrow\!\!\!\!\rightarrow (A/N)^s$ for $s \in \mathbb{N}_0$, and will pass through these isomorphisms often without mention. More generally, we will henceforth view $\mathrm{Inj}_N^{s,r}$ and $\mathrm{Sur}_N^{r,s}$ as maps between $(N^{-1}/A)^r$ and $(N^{-1}/A)^s$ instead of between $(A/N)^r$ and $(A/N)^s$, and similarly for Free_N^r, $\mathrm{Free}_N^{s,r}$ and their injective and surjective selections.

Analogously to Proposition 4.12, we can view $\overleftarrow{\mathcal{L}_N^r}$ as a disjoint union of $\mathcal{L}_N^s$ for $0 \le s \le r$, but on the contrary, we usually have multiple copies of $\mathcal{L}_N^s$.

4.52 Theorem. *If δ is an injective selection of* Free_N^r*, then we have a bijection*

$$\bigsqcup_{\substack{0\le s\le r\\ U\in\mathrm{Free}_N^{s,r}}} \mathcal{L}_N^s \xrightarrow{\sim} \overleftarrow{\mathcal{L}_N^r}$$
$$(\Lambda,\alpha)_U \mapsto (\Lambda, \delta_U \circ \alpha^{-1}).$$

Proof. We show that the above map from the union over the $\mathrm{Free}_N^{s,r}$ to $\overleftarrow{\mathcal{L}_N^r}$ is a bijection by describing its inverse: let $(\Lambda,\iota) \in \overleftarrow{\mathcal{L}_N^r}$ with Λ of rank s. Then $\operatorname{Im}\iota$ is free of rank s via similar arguments as before, and ι and $\delta_{\operatorname{Im}\iota}$ induce isomorphisms $\iota' : N^{-1}\Lambda/\Lambda \hookrightarrow\!\!\!\!\rightarrow \operatorname{Im}\iota$ and $\delta'_{\operatorname{Im}\iota} : (N^{-1}/A)^s \hookrightarrow\!\!\!\!\rightarrow \operatorname{Im}\iota$; hence we obtain an s-level N structure $\alpha_\iota = \iota'^{-1}\circ\delta'_{\operatorname{Im}\iota}$ for Λ which satisfies $\delta_{\operatorname{Im}\iota}\circ\alpha_\iota^{-1} = \iota$. Call the map given in the theorem statement by the name 'tog' and this proposed inverse by the name 'apt'[‡], so that $\mathrm{tog}\,(\Lambda,\alpha)_U = (\Lambda, \delta_U\circ\alpha^{-1})$ and $\mathrm{apt}\,(\Lambda,\iota) = (\Lambda,\alpha_\iota)_{\operatorname{Im}\iota}$. Then

$$\begin{aligned}
\mathrm{tog}\,\mathrm{apt}\,(\Lambda,\iota) &= \mathrm{tog}\,(\Lambda,\alpha_\iota)_{\operatorname{Im}\iota} = (\Lambda, \delta_{\operatorname{Im}\iota}\circ\alpha_\iota^{-1}) = (\Lambda,\iota) \quad \text{and}\\
\mathrm{apt}\,\mathrm{tog}\,(\Lambda,\alpha)_U &= \mathrm{apt}\,(\Lambda,\delta_U\circ\alpha^{-1}) = (\Lambda, \alpha_{\delta_U\circ\alpha^{-1}})_{\operatorname{Im}(\delta_U\circ\alpha^{-1})}\\
&= (\Lambda, (\delta_U\circ\alpha^{-1})'^{-1}\circ\delta'_{\operatorname{Im}\delta_U})_{\operatorname{Im}\delta_U} = (\Lambda,(\delta_U\circ\alpha^{-1})'^{-1}\circ\delta'_U)_U\\
&= (\Lambda,\alpha)_U,
\end{aligned}$$

which completes the proof. □

[‡]Short for 'together' and 'apart', respectively.

4.53 Definition. In the above decomposition of $\overleftarrow{\mathcal{L}^r_N}$ into a disjoint union of $\mathcal{L}^s_N$ for $0 \leq s \leq r$, each of the uniands[§] $\mathcal{L}^s_N$ is called a *stratum of dimension* s and *codimension* $r - s$; if $s < r$, they are also called *boundary strata*, and in the case where $s = r$ the stratum $\mathcal{L}^r_N$ is called the *main stratum*.

We can now combine the decomposition of $\overleftarrow{\mathcal{L}^r_N}$ into strata $\mathcal{L}^s_N$ for $0 \leq s \leq r$ with the identification of each stratum as a double quotient:

4.54 Proposition. *If δ is an injective selection of Free^r_N and for $0 \leq s \leq r$ we let $K_s(N) = \ker\big(\mathrm{GL}_s(\hat{A}) \twoheadrightarrow \mathrm{GL}_s(A/N)\big)$, then there is a bijection*

$$\bigsqcup_{\substack{0 \leq s \leq r \\ U \in \mathrm{Free}^{s,r}_N}} \mathrm{GL}_s(F)\backslash\big(\Psi^s \times \mathrm{GL}_s(\mathbb{A}^{fin}_F)/K_s(N)\big) \quad \xrightarrow{\sim} \quad \overleftarrow{\mathcal{L}^r_N}$$

which sends the equivalence class of a pair $(\psi, g)_U \in \Psi^s \times \mathrm{GL}_s(\mathbb{A}^{fin}_F)$ in the uniand corresponding to $U \in \mathrm{Free}^{s,r}_N$ to the lattice $\Lambda = \psi\big(F^s \cap g\hat{A}^s\big)$ and the r-inverse level structure ι which makes the following diagram commute:

$$\begin{tikzcd}
(N^{-1}/A)^s \arrow[r, hook, "\delta_U"] \arrow[d, hook, "\subset"'] & (N^{-1}/A)^r & & N^{-1}\Lambda/\Lambda \arrow[ll, hook', "\iota"'] \\
N^{-1}\hat{A}^s/\hat{A}^s \arrow[r, hook, two heads, "g"'] & N^{-1}g\hat{A}^s/g\hat{A}^s & & N^{-1}\big(F^s \cap g\hat{A}^s\big)/\big(F^s \cap g\hat{A}^s\big) \arrow[ll, hook', two heads, "\subset"] \arrow[u, hook, two heads, "\psi"']
\end{tikzcd}$$

Proof. This is a simple composition of Theorems 3.19 and 4.52. □

The number of strata

4.55 Corollary. *The number of strata of dimension s in $\overleftarrow{\mathcal{L}^r_N}$ is*

$$\frac{\#\mathrm{GL}_r(A/N)}{\#\mathrm{GL}_s(A/N)\#\mathrm{GL}_{r-s}(A/N)\#(A/N)^{s(r-s)}}.$$

Proof. The number in question is the number of free submodules of $(A/N)^r$ of rank s, or equivalently the cardinality of $\mathrm{Inj}^{s,r}_N/\mathrm{GL}_s(A/N)$. Now by Proposition 4.44 the right action of $\mathrm{GL}_s(A/N)$ on $\mathrm{Inj}^{s,r}_N$ is free; hence the cardinality of $\mathrm{Inj}^{s,r}_N/\mathrm{GL}_s(A/N)$ is the ratio of the cardinalities of $\mathrm{Inj}^{s,r}_N$ and $\mathrm{GL}_s(A/N)$, and so Corollary 4.47 completes the proof. □

[§] Uniands are to unions as summands are to sums.

Interestingly, the above number is invariant under the involution $s \mapsto r - s$.

4.56 Definition. For nonnegative integer r and an ideal N of A which factorises into a product of prime ideals as $N = \prod_i \mathfrak{p}_i^{d_i}$, we define the function

$$\phi^r(N) = |N|^r \cdot \prod_i \left(1 - |\mathfrak{p}_i|^{-r}\right)$$

analogously to the definition of the classical Euler ϕ function.

There should be no confusion between this use of ϕ^r as a function on ideals and ϕ^Λ as a polynomial, since r is an integer while Λ is a lattice.Note that ϕ^r is a multiplicative function, i.e. if M and N are coprime ideals (i.e. $M + N = A$) then $\phi^r(MN) = \phi^r(M)\phi^r(N)$.

4.57 Proposition.

$$\#\mathrm{GL}_r(A/N) = |N|^{r(r-1)/2} \cdot \phi^r(N)\phi^{r-1}(N) \cdots \phi^1(N).$$

This has been proven in [Bre10, Lemma 2.3]; we include the following proof

Proof. By the Chinese Remainder Theorem, $A/N \simeq \prod_i A/\mathfrak{p}_i^{d_i}$; in fact, we have $\mathrm{GL}_r(A/N) \simeq \prod_i \mathrm{GL}_r\left(A/\mathfrak{p}_i^{d_i}\right)$. So we calculate $\#\mathrm{GL}_r(A/N)$ for the case when $N = \mathfrak{p}^d$, and multiply the results together at the end.

So consider the elements of $\mathrm{GL}_r(A/\mathfrak{p}^d)$, which we consider as $r \times r$ matrices of elements in $A/\mathfrak{p}^d$ with determinant in $(A/\mathfrak{p}^d)^\times$, using the canonical basis for $(A/\mathfrak{p}^d)^r$. We also consider the canonical mod-$\mathfrak{p}$ projection $\pi : A/\mathfrak{p}^d \twoheadrightarrow A/\mathfrak{p}$, with $A/\mathfrak{p}$ being a field, and we use the same symbol π for the projection $\mathrm{GL}_r(A/\mathfrak{p}^d) \twoheadrightarrow \mathrm{GL}_r(A/\mathfrak{p})$. Note that for an $r \times r$ matrix $\gamma \in M_{r\times r}(A/\mathfrak{p}^d)$, $\det\gamma \in (A/\mathfrak{p}^d)^\times \iff \pi(\det\gamma) = \det\pi(\gamma) \in (A/\mathfrak{p})^\times$.

Now it is well known that for a finite field F, the cardinality of $\mathrm{GL}_r(F)$ is

$$\#\mathrm{GL}_r(F) = (\#(F)^r - 1)(\#(F)^r - \#(F)) \cdots \left(\#(F)^r - \#(F)^{r-1}\right).$$

Also, the number of $\gamma \in M_{r\times r}(A/\mathfrak{p}^d)$ such that $\pi(\gamma) = \gamma'$ for a given $\gamma' \in M_{r\times r}(A/\mathfrak{p})$ is equal to $\left(\#(A/\mathfrak{p}^d)/\#(A/\mathfrak{p})\right)^{r\times r} = |\mathfrak{p}|^{(d-1)r^2}$. Hence, since $A/\mathfrak{p}$ is a field of cardinality $\#(A/\mathfrak{p}) = |\mathfrak{p}|$, we have that

$$\begin{aligned}\#\mathrm{GL}_r\left(A/\mathfrak{p}^d\right) &= \#(A/\mathfrak{p})^{(d-1)r^2}\#\mathrm{GL}_r(A/\mathfrak{p}) \\ &= |\mathfrak{p}|^{(d-1)r^2}(|\mathfrak{p}|^r - 1)\cdots\left(|\mathfrak{p}|^r - |\mathfrak{p}|^{r-1}\right) \\ &= |\mathfrak{p}|^{dr^2}\left(1 - |\mathfrak{p}|^{-r}\right)\cdots\left(1 - |\mathfrak{p}|^{-1}\right).\end{aligned}$$

Thus as desired:

$$\begin{aligned}
\#\mathrm{GL}_r(A/N) &= \prod_i \#\mathrm{GL}_r\big(A/\mathfrak{p}_i^{d_i}\big) \\
&= \prod_i |\mathfrak{p}_i|^{d_i r^2}\big(1-|\mathfrak{p}_i|^{-r}\big)\cdots\big(1-|\mathfrak{p}_i|^{-1}\big) \\
&= |N|^{r^2}\prod_i \big(1-|\mathfrak{p}_i|^{-r}\big)\cdots\big(1-|\mathfrak{p}_i|^{-1}\big) \\
&= |N|^{r(r-1)/2}\cdot\phi^r(N)\phi^{r-1}(N)\cdots\phi^1(N).
\end{aligned}$$

□

4.58 Corollary. *The number of strata of $\overleftarrow{\mathcal{L}^r_N}$ of dimension s is equal to*

$$\frac{\phi^r(N)\cdots\phi^1(N)}{\phi^s(N)\cdots\phi^1(N)\cdot\phi^{r-s}(N)\cdots\phi^1(N)}.$$

Proof. Apply Proposition 4.57 to Corollary 4.55. □

Interestingly, this number is multiplicative, i.e. for coprime ideals N, M we have that the number of strata of dimension s of $\overleftarrow{\mathcal{L}^r_{MN}}$ is equal to the product of those of $\overleftarrow{\mathcal{L}^r_M}$ and $\overleftarrow{\mathcal{L}^r_N}$. Also, this formula is very reminiscent of the formula for binomial coefficients in terms of factorials.

4.59 Note that if we set $N = A$ all through this and the previous section, each lattice Λ has exactly one (r-inverse) level N structure, being the zero map. Hence we have the following isomorphisms between spaces: $\mathcal{L}^r_A \simeq \mathcal{L}^r$ and $\overleftarrow{\mathcal{L}^r_A} \simeq \mathcal{L}^{\leq r}$; and so considerations of inclusion of level structure do not exclude the spaces without level structure, as long as $N \neq A$ is not assumed.

The action of $\mathrm{GL}_r(A/N)$ on $\overleftarrow{\mathcal{L}^r_N}$

Recall the action of $\gamma \in \mathrm{GL}_r(A/N)$ on $\mathcal{L}^r_N$ defined by $\gamma(\Lambda,\alpha) = (\Lambda, \alpha\circ\gamma^{-1})$. Given the identification of the rank-r subset of $\overleftarrow{\mathcal{L}^r_N}$ with $\mathcal{L}^r_N$ in Paragraph 4.20 via $(\Lambda,\iota) \leftrightarrow (\Lambda,\alpha_\iota) = (\Lambda,\iota^{-1})$, we see how to extend this action to $\overleftarrow{\mathcal{L}^r_N}$:

4.60 Definition. $\mathrm{GL}_r(A/N)$ acts on $\overleftarrow{\mathcal{L}^r_N}$ on the left via $\gamma(\Lambda,\iota) = (\Lambda, \gamma\circ\iota)$.

4.61 Proposition. *The action of* $\mathrm{GL}_r(A/N)$ *on* $\overleftarrow{\mathcal{L}^r_N}$ *is an isometry.*

Proof. Let $(\Lambda_1, \iota_1), (\Lambda_2, \iota_2) \in \overleftarrow{\mathcal{L}^r_N}$ and $\gamma \in \mathrm{GL}_r(A/N)$.

Note that for any $(\Lambda, \iota) \in \overleftarrow{\mathcal{L}^r_N}$, $\mathrm{Im}(\gamma \circ \iota) = \gamma \operatorname{Im} \iota$ so that

$$\begin{aligned}
\mu_{\Lambda,\gamma\iota}(\gamma l) &= \begin{cases} e_\Lambda(\iota^{-1}\gamma^{-1}(\gamma l))^{-1} & \text{if } \gamma l \in \mathrm{Im}(\gamma\iota) - \{0\} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} e_\Lambda(\iota^{-1}(l))^{-1} & \text{if } l \in \operatorname{Im}\iota - \{0\} \\ 0 & \text{otherwise} \end{cases} \\
&= \mu_{\Lambda,\iota}(l).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{d}_{\overleftarrow{\mathcal{L}^r_N}}\big(\gamma(\Lambda_1, \iota_1), \gamma(\Lambda_2, \iota_2)\big) &= \mathrm{d}_{\overleftarrow{\mathcal{L}^r_N}}\big((\Lambda_1, \gamma\iota_1), (\Lambda_2, \gamma\iota_2)\big) \\
&= \mathrm{d}_{\mathcal{L}}(\Lambda_1, \Lambda_2) + \sum_{l \in (N^{-1}/A)^r} |\mu_{\Lambda_1,\gamma\iota_1}(l) - \mu_{\Lambda_2,\gamma\iota_2}(l)| \\
&= \mathrm{d}_{\mathcal{L}}(\Lambda_1, \Lambda_2) + \sum_{l \in (N^{-1}/A)^r} |\mu_{\Lambda_1,\gamma\iota_1}(\gamma l) - \mu_{\Lambda_2,\gamma\iota_2}(\gamma l)| \\
&= \mathrm{d}_{\mathcal{L}}(\Lambda_1, \Lambda_2) + \sum_{l \in (N^{-1}/A)^r} |\mu_{\Lambda_1,\iota_1}(l) - \mu_{\Lambda_2,\iota_2}(l)| \\
&= \mathrm{d}_{\overleftarrow{\mathcal{L}^r_N}}\big((\Lambda_1, \iota_1), (\Lambda_2, \iota_2)\big)
\end{aligned}$$

as desired. □

We now see how this action interacts with the decomposition of $\overleftarrow{\mathcal{L}^r_N}$ into strata of the form $\mathcal{L}^s_N$:

4.62 Proposition. *For δ an injective selection of Free^r_N, $\gamma \in \mathrm{GL}_r(A/N)$ acts on the decomposition $\bigsqcup_{\substack{0 \le s \le r \\ U \in \mathrm{Free}^{s,r}_N}} \mathcal{L}^s_N$ of $\overleftarrow{\mathcal{L}^r_N}$ by*

$$\gamma(\Lambda, \alpha)_U = (\Lambda, \alpha \circ \delta_U^{-1} \circ \gamma^{-1} \circ \delta_{\gamma U})_{\gamma U}.$$

Proof. Let $\gamma(\Lambda, \alpha)_U = (\Lambda', \alpha')_{U'} \mapsto (\Lambda', \delta_{U'} \circ \alpha'^{-1})$. Firstly, it is easy to see that $\Lambda' = \Lambda$. Secondly, we have from Definition 4.60 that

$$(\Lambda', \alpha')_{U'} = \gamma(\Lambda, \alpha)_U \mapsto \gamma(\Lambda, \delta_U \circ \alpha^{-1}) = (\Lambda, \gamma \circ \delta_U \circ \alpha^{-1}).$$

Hence $U' = \mathrm{Im}(\gamma \circ \delta_U \circ \alpha^{-1}) = \gamma \operatorname{Im}(\delta_U) = \gamma U$. Finally,

$$\begin{aligned}
\delta_{U'} \circ \alpha'^{-1} = \gamma \circ \delta_U \circ \alpha^{-1} &\iff \alpha'^{-1} = \delta_{\gamma U}^{-1} \circ \gamma \circ \delta_U \circ \alpha^{-1} \\
&\iff \alpha' = \alpha \circ \delta_U^{-1} \circ \gamma^{-1} \circ \delta_{\gamma U}.
\end{aligned}$$ □

Note that in the above action, although δ_U is not always a bijection and so δ_U^{-1} does not exist as a function from $(N^{-1}/A)^r$ to U, since

$$\operatorname{Im}(\gamma^{-1} \circ \delta_{\gamma U}) = \gamma^{-1} \operatorname{Im} \delta_{\gamma U} = \gamma^{-1}\gamma U = U$$

the composite $\delta_U^{-1} \circ \gamma^{-1} \circ \delta_{\gamma U} : \gamma U \hookrightarrow\!\!\!\!\rightarrow U$ is in fact always well defined.

Also note that if for the main stratum $\mathcal{L}_N^r$ of $\overleftarrow{\mathcal{L}_N^r}$ (which corresponds to $U = (N^{-1}/A)^r$) we have that δ_U is the identity map, then the action of γ on the main stratum simplifies to $\gamma(\Lambda, \alpha)_U = (\Lambda, \alpha \circ \gamma^{-1})_U$, i.e. the way the action is defined on $\mathcal{L}_N^r$.

The action of $\left(\mathbb{A}_F^{fin}\right)^\times$

Recall the action of $x \in \left(\mathbb{A}_F^{fin}\right)^\times$ on $\mathcal{L}_N^r$ given by $x(\Lambda, \alpha) = (J(x)^{-1}\Lambda, x^{-1} \circ \alpha)$ in Proposition 3.54. The action of $\left(\mathbb{A}_F^{fin}\right)^\times$ on $\mathcal{L}_N^r$ induces a rigid analytic action of the set of fractional ideals on $\mathcal{L}^r$ by $\Lambda \overset{J}{\mapsto} J^{-1}\Lambda$. In fact, we can extend this action to $\mathcal{L}^{\leq r}$ by the following result:

4.63 Proposition. *The action of $\mathcal{J}(A)$ on $\mathcal{L}^{\leq r}$ given by $J(\Lambda) = J^{-1}\Lambda$ is a homeomorphism.*

Proof. Since $(J^{-1})^{-1}J^{-1}\Lambda = \Lambda$ for any lattice Λ and fractional ideal J, it is enough to show that each action by J is continuous, since then the inverse action by J^{-1} is continuous too. To this end let $\Lambda' \in \mathcal{L}^{\leq r}$ be a variable lattice with $\Lambda' \to \Lambda$, so that $\mathrm{d}_{\mathcal{L}}(\Lambda', \Lambda) = \sup_{|z|\leq 1}|e_{\Lambda'}(z) - e_\Lambda(z)| \to 0$, and consider first the case of J being a (non-fractional) ideal of A. Then

$$\begin{aligned}
&\mathrm{d}_{\mathcal{L}}(J^{-1}\Lambda', J^{-1}\Lambda)\\
&= \sup_{|z|\leq 1}|e_{J^{-1}\Lambda'}(z) - e_{J^{-1}\Lambda}(z)|\\
&= \sup_{|z|\leq 1}\left|\phi_J^{\Lambda'}(e_{\Lambda'}(z)) - \phi_J^{\Lambda}(e_\Lambda(z))\right|\\
&\leq \sup_{|z|\leq 1}\left|\phi_J^{\Lambda'}(e_{\Lambda'}(z)) - \phi_J^{\Lambda}(e_{\Lambda'}(z))\right| + \sup_{|z|\leq 1}\left|\phi_J^{\Lambda}(e_{\Lambda'}(z) - e_\Lambda(z))\right|.
\end{aligned}$$

Now by Proposition 5.35, each of the coefficients of the polynomial $\phi_J^{\Lambda'}(X)$ of degree at most X^{q^r} is continuous on $\mathcal{L}^{\leq r}$ and hence $\phi_J^{\Lambda'}$ is uniformly continuous on bounded sets. As $\Lambda' \to \Lambda$, $e_{\Lambda'(z)}$ is bounded on $|z| \leq 1$; thus the first term goes to zero. And since ϕ_J^{Λ} is a polynomial with zero constant term, the second term also goes to zero; thus the action of J is continuous.

We now consider the case of J being the reciprocal of a principal ideal, i.e. $J = (a)^{-1}$ for $a \in A$. Here we have that

$$\begin{aligned}
\mathrm{d}_{\mathcal{L}}(J^{-1}\Lambda', J^{-1}\Lambda) &= \mathrm{d}_{\mathcal{L}}(a\Lambda', a\Lambda) \\
&= \sup_{|z|\leq 1} |e_{a\Lambda'}(z) - e_{a\Lambda}(z)| \\
&\leq \sup_{|z|\leq |a|} |e_{a\Lambda'}(z) - e_{a\Lambda}(z)| \\
&= \sup_{|z|\leq 1} |e_{a\Lambda'}(az) - e_{a\Lambda}(az)| \\
&= \sup_{|z|\leq 1} |ae_{\Lambda'}(z) - ae_{\Lambda}(z)| \\
&= |a| \, \mathrm{d}_{\mathcal{L}}(\Lambda', \Lambda) \\
&\to 0.
\end{aligned}$$

Finally, a general fractional ideal J can be written as $J = (a)^{-1}J'$ for some $a \in A$ and a non-fractional ideal J'; since the actions of $(a)^{-1}$ and J' are continuous, the action of J is too. $\square$

Unfortunately we have not yet, as of the time of writing, been able to extend the action of $\left(\mathbb{A}_F^{fin}\right)^{\times}$ to $\overleftarrow{\mathcal{L}_N^r}$ in a sufficiently 'nice' way. In fact, we suspect this extension to not be possible, although we have not yet proven this either.

5 Modular Forms

Definitions

Modular forms on $\mathcal{L}_N^r$

To understand the structure and shape of the space of lattices with level structure or isomorphism classes of Drinfeld modules with level structure, one strategy is to investigate the collections of functions on those spaces.

5.1 Definition. For $k \in \mathbb{Z}$, a *weak modular form* f of *weight* k and *rank* r for the congruence subgroup $K(N)$ is a function $\mathcal{L}_N^r \to \mathbb{C}_\infty$ which is

- holomorphic*, and
- homogeneous of degree $-k$, i.e.

$$f(t \cdot \Lambda, t \cdot \alpha) = t^{-k} f(\Lambda, \alpha) \quad \text{for all } (\Lambda, \alpha) \in \mathcal{L}_N^r \text{ and } t \in \mathbb{C}_\infty^\times.$$

We denote the $\mathbb{C}_\infty$-vector space of weak modular forms for $K(N)$ of weight k and rank r by $\mathbf{Weak}_N^{k,r}$.

5.2 It is easy to check that for $t_1, t_2 \in \mathbb{C}_\infty$ and weak modular forms $f_1, f_2 \in \mathbf{Weak}_N^{k,r}$, we have that $t_1 f_1 + t_2 f_2 \in \mathbf{Weak}_N^{k,r}$, so that $\mathbf{Weak}_N^{k,r}$ is a $\mathbb{C}_\infty$-vector space. Moreover, the product of two modular forms of weight k_1 and k_2 is a modular form of weight $k_1 + k_2$, so that $\mathbf{Weak}_N^r := \oplus_{k=0}^\infty \mathbf{Weak}_N^{k,r}$ is a graded $\mathbb{C}_\infty$-algebra, graded by the weight k.

5.3 Definition. A *(strong) modular form* of *weight* k and *rank* r for $K(N)$ is a function $\overleftarrow{\mathcal{L}_N^r} \to \mathbb{C}_\infty$ which is:

- continuous on $\overleftarrow{\mathcal{L}_N^r}$,
- homogeneous of degree $-k$, and

*Here holomorphic on $\mathcal{L}_N^r$ means holomorphic on the isomorphic rigid analytic space $\mathrm{GL}_r(F) \Big\backslash \left(\Psi^r \times \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right) \Big/ K(N) \right)$, i.e. holomorphic on the space $\Psi^r \subset \mathbb{C}_\infty^r$ and invariant under the two group actions of $\mathrm{GL}_r(F)$ and $K(N)$.

- holomorphic on the main stratum $\mathcal{L}_N^r$ of $\overleftarrow{\mathcal{L}_N^r}$.

In general we will omit the adjective *strong*, unless we are comparing strong with weak modular forms, and will omit the reference to $K(N)$.

We denote the $\mathbb{C}_\infty$-vector space of strong modular forms for $K(N)$ of weight k and rank r by $\mathbf{Strong}_N^{k,r}$.

5.4 As in Paragraph 5.2, the strong modular forms of rank r form a graded $\mathbb{C}_\infty$-algebra $\mathbf{Strong}_N^r := \bigoplus_{k=0}^{\infty} \mathbf{Strong}_N^{k,r}$, graded by the weight k.

5.5 It is apparent that the restriction of a strong modular form to the main stratum $\mathcal{L}_N^r$ of $\overleftarrow{\mathcal{L}_N^r}$ is a weak modular form, and given the denseness of $\mathcal{L}_N^r$ in $\overleftarrow{\mathcal{L}_N^r}$, the values of a strong modular form (which is continuous) on the boundary strata can be recovered from the values on the main stratum. However, not every weak modular form can necessarily be extended to a strong modular form, if for instance it does not have a limit as one tends to the boundary of $\overleftarrow{\mathcal{L}_N^r}$. Thus there is a injection $\mathbf{Strong}_N^{k,r} \hookrightarrow \mathbf{Weak}_N^{k,r}$ of $\mathbb{C}_\infty$-vector spaces for each weight k and an injection $\mathbf{Strong}_N^r \hookrightarrow \mathbf{Weak}_N^r$ of graded $\mathbb{C}_\infty$-algebras.

5.6 Proposition. *The only modular forms of weight* 0 *are the constant maps, and the only modular forms of negative weight are the zero maps.*

Proof. Let f be a modular form of weight 0, $(\Lambda, \iota) \in \overleftarrow{\mathcal{L}_N^r}$, and $t \in \mathbb{C}_\infty^\times$ be large. Then since f is homogeneous of weight 0, $f(\Lambda, \iota) = f(t\Lambda, \iota t)$. But since f is continuous, and $(t\Lambda, \iota t) \to 0$ as $|t| \to \infty$, we get that $f(\Lambda, \iota) = f(0)$; thus f is constant. Conversely, any constant map is a modular form of weight 0.

Now let f be a modular form of weight $k < 0$, let $(\Lambda, \iota) \in \overleftarrow{\mathcal{L}_N^r}$, and let $t \in \mathbb{C}_\infty^\times$ be large. Then $f(\Lambda, \iota) = t^k f(t\Lambda, t\iota)$; but $(t\Lambda, t\iota) \to 0$ and $t^k \to 0$ as $|t| \to \infty$, so since f is continuous we get that $t^k f(t\Lambda, t\iota) \to 0 \times f(0) = 0$; thus f is the zero map, which is a modular form of weight k. $\square$

5.7 Definition. A modular form $f \in \mathbf{Strong}_N^{k,r}$ is called a *cusp form* if $f(\Lambda, \iota) = 0$ for (Λ, ι) in the boundary strata, i.e. when Λ is of rank strictly less than r.

The $\mathbb{C}_\infty$-vector space of cusp forms of weight k and rank r for $K(N)$ is denoted $\mathbf{Cusp}_N^{k,r}$, and the $\mathbf{Strong}_N^r$-algebra of all cusp forms for $K(N)$, which is also a graded $\mathbb{C}_\infty$-algebra graded by weight, is denoted $\mathbf{Cusp}_N^r$.

$\mathbf{Cusp}_N^r$ is also an ideal of $\mathbf{Strong}_N^r$.

The left action of $\mathrm{GL}_r(A/N)$ on $\overleftarrow{\mathcal{L}^r_N}$ carries over to a right action on the space of modular forms:

5.8 Definition. For a modular form f and $\gamma \in \mathrm{GL}_r(A/N)$, we define the function $f|\gamma$ on $\overleftarrow{\mathcal{L}^r_N}$ by $(f|\gamma)(\Lambda, \iota) = f(\gamma(\Lambda, \iota)) = f(\Lambda, \gamma \circ \iota)$.

5.9 Proposition. *The map $f \mapsto f|\gamma$ is a right action of $\gamma \in \mathrm{GL}_r(A/N)$ on* $\mathbf{Strong}^r_N$*, which preserves the weight k and maps cusp forms to cusp forms.*

Proof. Let f be a modular form of weight k; then since the action of $\gamma \in \mathrm{GL}_r(A/N)$ is an isometry, $f|\gamma$ is also continuous; since the action is a rigid analytic automorphism of $\mathcal{L}^r_N$, $f|\gamma$ is also holomorphic on $\mathcal{L}^r_N$; and it is easy to see that $f|\gamma$ is also homogeneous of degree $-k$. Thus γ maps $\mathbf{Strong}^{k,r}_N$ to $\mathbf{Strong}^{k,r}_N$, and it is easy to see that $f \mapsto f|\gamma$ satisfies the conditions of a right action.

Finally, since the action of γ leaves Λ unchanged, if $f(\Lambda, \iota) = 0$ for Λ of rank less than r, then the same is true for $f|\gamma$. $\square$

Modular forms for $\mathcal{L}^{\leq r}$

5.10 Definition. For $k \in \mathbb{Z}$, a modular form of weight k and rank r for $\mathcal{L}^{\leq r}$ is a function $\mathcal{L}^{\leq r} \to \mathbb{C}_\infty$ which is:

- continuous on $\mathcal{L}^{\leq r}$,
- homogeneous of degree $-k$, and
- holomorphic on the main stratum $\mathcal{L}^r$ of $\mathcal{L}^{\leq r}$.

We denote the space of strong modular forms of weight k and rank r for $\mathcal{L}^{\leq r}$ by $\mathbf{Strong}^{k,r}$.

5.11 As in Paragraph 5.2, the modular forms on $\mathcal{L}^{\leq r}$ of rank r form a graded $\mathbb{C}_\infty$-algebra $\mathbf{Strong}^r := \bigoplus_{k=0}^{\infty} \mathbf{Strong}^{k,r}$, graded by the weight k.

5.12 Note that if a modular form f for $\overleftarrow{\mathcal{L}^r_N}$ is independent of the level structure ι (i.e. if $f(\Lambda, \iota_1) = f(\Lambda, \iota_2)$ for any two inverse level N structures ι_1, ι_2 for a lattice Λ, or equivalently it is invariant under the action of $\mathrm{GL}_r(A/N)$) then it induces a unique modular form on $\mathcal{L}^{\leq r} \simeq \mathrm{GL}_r(A/N)\big\backslash \overleftarrow{\mathcal{L}^r_N}$.

5.13 Definition. A modular form $f \in \mathbf{Strong}^{k,r}$ is called a *cusp form* if $f(\Lambda) = 0$ for Λ in the boundary strata $\mathcal{L}^{\leq r-1}$, i.e. Λ has rank less than r.

The $\mathbb{C}_\infty$-vector space of cusp forms of weight k and rank r is denoted $\mathbf{Cusp}^{k,r}$, and the $\mathbf{Strong}^r$-algebra of all cusp forms, which is also a graded $\mathbb{C}_\infty$-algebra graded by weight, is denoted $\mathbf{Cusp}^r$.

The action of $\left(\mathbb{A}_F^{fin}\right)^\times$ on $\mathcal{L}^{\leq r}$, or equivalently of $\mathcal{J}(A)$ on $\mathcal{L}^{\leq r}$, also carries over to an action on the space of modular forms on $\mathcal{L}^{\leq r}$:

5.14 Definition. For a modular form f on $\mathcal{L}^{\leq r}$ and a fractional ideal J, we define the function $f|J$ on $\mathcal{L}^{\leq r}$ by $(f|J)(\Lambda) \coloneqq f(J(\Lambda)) = f(J^{-1}\Lambda)$.

5.15 Proposition. *The map $f \mapsto f|J$ is an action of $\mathcal{J}(A)$ on the space of modular forms f weight k and rank r on $\mathcal{L}^{\leq r}$ for each $k \in \mathbb{Z}$, which maps cusp forms to cusp forms.*

Proof. Let f be a modular form of weight k; then since the action of J is a homeomorphism, $f|J$ is also continuous; since the action is a rigid analytic automorphism of $\mathcal{L}^r$, $f|J$ is also holomorphic on $\mathcal{L}^r$; and since the action of J commutes with scaling of a lattice, $f|J$ is also homogeneous of degree $-k$. Thus J maps the space of modular forms on $\mathcal{L}^{\leq r}$ of weight k to itself.

Finally, if Λ has rank less than r then so does $J^{-1}\Lambda$, so that if f is a cusp form then $f|J$ is too. □

5.16 As a special case of this action by fractional ideals, we have the instances where $J = (t)$ is a principal ideal for $t \in F^\times$. Here, J is simply scaling the lattices by t^{-1}, and thus is scaling the $\mathbf{Strong}_N^{k,r}$ by t^k. So the more interesting case is of non-principal ideals, with the ideal class group $\mathrm{Cl}(F) = \mathcal{J}(A)/\mathrm{Prin}(A)$ being of interest.

Examples of modular forms

In this section we list some classes of examples of modular forms. There is large overlap between these modular forms and the examples given in [BBP3], the correspondence between which may be made clearer by the connective results in the following section. Before we start our listing, a lemma:

5.17 Lemma. *Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be a series which converges on $|z| \leq 1$. Then if $|f(z)| < \epsilon$ for all z with $|z| = 1$, each coefficient f_k has $|f_k| < \epsilon$.*

Proof. Let $\ell \in \mathbb{N}_0$. Since $f(z)$ converges, we have that $f_k \to 0$ as $k \to \infty$; so let $|f_k| < \epsilon$ for $k \geq K_\epsilon$. Since $\mathbb{C}_\infty \supset \overline{\mathbb{F}_q}$, we can let $\zeta \in \mathbb{F}_{q^n} - \mathbb{F}_{q^{n-1}} \subset \mathbb{C}_\infty$ be a primitive N-th root of unity, for some n with $N = q^n - 1 > K_\epsilon$.

Now consider the sum

$$F = \sum_{i=0}^{N-1} f(\zeta^i)\zeta^{-i\ell} = \sum_{k=0}^{\infty} f_k \sum_{i=0}^{N-1} \zeta^{i(k-\ell)} = \sum_{\substack{k\in\mathbb{N}_0\\ N|k-\ell}} N f_k = -\sum_{k=0}^{\infty} f_{\ell+Nk}.$$

On the one hand, $|F| \le \max_{i=0}^{N-1}|f(\zeta^i)\zeta^{-i\ell}| = \max_{i=0}^{N-1}|f(\zeta^i)| < \epsilon$. On the other hand, $|F + f_\ell| \le \max_{k=1}^{\infty}|f_{\ell+Nk}| < \epsilon$; thus $|f_\ell| < \epsilon$ as desired. □

Eisenstein Series and e_Λ coefficient forms

5.18 Definition. For $k \in \mathbb{N}$ we define the *Eisenstein series*

$$E^k(\Lambda) = \sum_{\lambda\in\Lambda}{}' \lambda^{-k}.$$

These converge since the lattice elements go to infinity. Moreover, E^k is homogeneous of degree $-k$; i.e. $E^k(c\Lambda) = c^{-k}E^k(\Lambda)$ for $c \in \mathbb{C}_\infty^\times$.

There is a connection between these Eisenstein series and the Taylor expansion coefficients of e_Λ, or more specifically of $1/e_\Lambda$:

5.19 Proposition. *For* $|z| < \min'_{\lambda\in\Lambda}|\lambda|$,

$$\frac{1}{e_\Lambda(z)} - \frac{1}{z} = -\sum_{k=1}^{\infty} E^k(\Lambda) z^{k-1}.$$

Proof. By Proposition 2.15,

$$\frac{1}{e_\Lambda(z)} - \frac{1}{z} = \sum_{\lambda\in\Lambda}{}' \frac{1}{z-\lambda} = \sum_{\lambda\in\Lambda}{}' \frac{-1}{\lambda}\frac{1}{1-z/\lambda} = -\sum_{\lambda\in\Lambda}{}'\sum_{k=1}^{\infty}\frac{z^{k-1}}{\lambda^k}$$
$$= -\sum_{k=1}^{\infty} z^{k-1}\sum_{\lambda\in\Lambda}{}' \lambda^{-k} = -\sum_{k=1}^{\infty} E^k(\Lambda) z^{k-1}.$$ □

5.20 Lemma. *For each* $k \in \mathbb{N}$ *and rank* r*, the Eisenstein series* $E^k(\Lambda)$ *is a holomorphic function of* Λ *on* $\mathcal{L}^r$.

Proof. We show that $E^k(\Lambda)$ is a holomorphic function on $\mathcal{L}^r_N$; the fact that it has no dependence on level structure then implies holomorphicity on $\mathcal{L}^r$.

Recall the isomorphism $\mathcal{L}^r_N \simeq \mathrm{GL}_r(F)\backslash\big((\Omega^r \times \mathbb{C}^\times_\infty) \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big)$, and consider E^k firstly as a function on $((\omega, \psi_r), g) \in (\Omega^r \times \mathbb{C}^\times_\infty) \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$. Since the topology on $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$ is discrete, we need only consider how $E^k(\Lambda)$ changes as ω and ψ_r vary; hence let g be constant, and define $\Lambda_g = F^r \cap g\hat{A}^r$, a strongly discrete 'lattice' in F^r. Then since

$$\Lambda = \kappa(\omega, \psi_r)\big(F^r \cap g\hat{A}^r\big) = \kappa(\omega, \psi_r)(\Lambda_g) \text{ with } \kappa(\omega, \psi_r) = \omega \cdot \frac{\psi_r}{\omega_r},$$

$$E^k(\Lambda) = \sum_{\lambda \in \Lambda} \lambda^{-k} = \sum_{\lambda \in \Lambda_g} \kappa(\omega, \psi_r)(\lambda)^{-k} = \psi_r^{-k} \sum_{\lambda \in \Lambda_g} [(\omega_1\lambda_1 + \cdots + \omega_r\lambda_r)/\omega_r]^{-k}.$$

Now ψ_r^{-k} is obviously holomorphic as a function of ψ_r, and for each $\lambda \in \Lambda_g$, the linear combination $(\omega_1\lambda_1 + \cdots + \omega_r\lambda_r)/\omega_r$ is a nonzero holomorphic function of ω, since the ω_i are F_∞-linearly independent. Since Λ_g is strongly discrete and $k > 0$, the above sum converges uniformly on affinoid subsets of $\Omega^r \times \mathbb{C}^\times_\infty$ and hence defines a holomorphic function.

Now since the actions of $\mathrm{GL}_r(F)$ and $K(N)$ leave the lattice Λ unchanged, $E^k(\Lambda)$ is also invariant under these actions, and hence is also holomorphic on $\mathcal{L}^r_N$, the quotient of $(\Omega^r \times \mathbb{C}^\times_\infty) \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$ by these actions. □

5.21 Proposition. *The coefficients $e_{\Lambda,i} = \big[z^{q^i}\big]e_\Lambda(z)$ for $i \in \mathbb{N}_0$ considered as functions of Λ are continuous on $\overleftarrow{\mathcal{L}^r_N}$ and homogeneous of degree $-q^i + 1$.*†

Proof. Homogeneity follows from the relation $e_{t\Lambda}(tz) = te_\Lambda(z)$ for $t \in \mathbb{C}^\times_\infty$. Also, note that $\mathrm{d}_{\overleftarrow{\mathcal{L}^r_N}}\big((\Lambda_1, \iota_1), (\Lambda_2, \iota_2)\big) < \epsilon \implies \sup_{|z|\le 1}|e_{\Lambda_1}(z) - e_{\Lambda_2}(z)| < \epsilon$. Thus by Lemma 5.17, $e_{\Lambda_1,i} - e_{\Lambda_2,i} = \big[z^{q^i}\big](e_{\Lambda_1}(z) - e_{\Lambda_2}(z)) < \epsilon$, whence follows the continuity. □

Now for the holomorphicity of these coefficient forms:

5.22 Proposition. *For $i \in \mathbb{N}_0$ and $k \in \mathbb{N}$ there are polynomials $G_i(x_1, x_2, \ldots)$ and $G^k(y_1, y_2, \ldots)$ with coefficients in $\mathbb{F}_p$ such that $e_{\Lambda,i} = G_i(E^1(\Lambda), E^2(\Lambda), \ldots)$ and $E^k(\Lambda) = G^k(e_{\Lambda,1}, e_{\Lambda,2}, \ldots)$.*

---

†Here and elsewhere, the notation $\big[z^k\big]f(z)$ denotes the coefficient of z^k in $f(z)$.

Proof. For $i \in \mathbb{N}_0$,

$$\begin{aligned}
e_{\Lambda,i} &= \left[z^{q^i}\right] e_\Lambda(z) \\
&= \left[z^{q^i}\right] z \left[1 - \left(1 - \frac{z}{e_\Lambda(z)}\right)\right]^{-1} \\
&= \left[z^{q^i - 1}\right] \sum_{n=0}^{\infty} \left(1 - \frac{z}{e_\Lambda(z)}\right)^n \\
&= \left[z^{q^i - 1}\right] \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} E^k(\Lambda) z^k\right)^n \qquad \text{by Proposition 5.19} \\
&= \sum_{n=0}^{q^i - 1} \left[z^{q^i - 1}\right] \left(\sum_{k=1}^{\infty} E^k(\Lambda) z^k\right)^n \\
&= \sum_{n=0}^{q^i - 1} \sum_{\substack{k_1 + \cdots + k_n = q^i - 1 \\ k_1, k_2, \ldots, k_n \geq 1}} E^{k_1}(\Lambda) E^{k_2}(\Lambda) \cdots E^{k_n}(\Lambda)
\end{aligned}$$

which is a polynomial in the $E^k(\Lambda)$ as desired. The second series of polynomials is found similarly, using a geometric series expansion on the relation

$$E^k(\Lambda) = \left[z^k\right] \left(1 - \frac{z}{e_\Lambda(z)}\right) = -\left[z^k\right] \frac{1}{1 + (e_\Lambda(z)/z - 1)}.$$ □

5.23 Corollary. *The coefficients $e_{\Lambda,i}$ are holomorphic on $\mathcal{L}^r$.*

Proof. Each $e_{\Lambda,i}$ is a polynomial in the $E^k(\Lambda)$ by Proposition 5.22, each of which is holomorphic on $\mathcal{L}^r$ by Lemma 5.20. □

5.24 Corollary. *The Eisenstein series $E^k(\Lambda)$ are continuous on $\overleftarrow{\mathcal{L}_N^r}$.*

Proof. Each $E^k(\Lambda)$ is a polynomial in the $e_{\Lambda,i}$ by Proposition 5.22, each of which is continuous on $\overleftarrow{\mathcal{L}_N^r}$ by Proposition 5.21. □

Thus the coefficients $e_{\Lambda,i}$ and the Eisenstein series $E^k(\Lambda)$ form our first two classes of modular forms for $\overleftarrow{\mathcal{L}_N^r}$, and are in fact modular forms on $\mathcal{L}^{\leq r}$. As such, we can extend these classes to more modular forms on $\mathcal{L}^{\leq r}$:

5.25 Proposition. *For each $J \in \mathcal{J}(A)$ and $i, k \in \mathbb{N}_0$, we have that $e_{J^{-1}\Lambda,i}$ and $E^k(J^{-1}\Lambda)$ are modular forms on $\mathcal{L}^{\leq r}$ of weight $q^i - 1$ and k respectively.*

Proof. $e_{J^{-1}\Lambda,i} = e_{\Lambda,i}|J$ and $E^k(J^{-1}\Lambda) = E^k(\Lambda)|J$, so Proposition 5.15 applies. □

Partial Eisenstein Series

5.26 Definition. For each $k \in \mathbb{N}$ and $l \in (N^{-1}/A)^r - \{0\}$, we define the *(partial) Eisenstein series* E_l^k on $\overleftarrow{\mathcal{L}_N^r}$ by

$$E_l^k(\Lambda, \iota) = \begin{cases} \sum_{v \in \iota^{-1}(l)} v^{-k} & \text{if } l \in \operatorname{Im} \iota \\ 0 & \text{otherwise} \end{cases}.$$

We will omit the adjective *partial*, unless contrasting with the *complete* Eisenstein series of Definition 5.18.

Restricting to the main stratum $\mathcal{L}_N^r$, we have the corresponding weak modular form:

5.27 Definition. For $k \in \mathbb{N}$ and $l \in (N^{-1}/A)^r - \{0\}$, we define the *weak (partial) Eisenstein series* E_l^k on $\mathcal{L}_N^r$ by

$$E_l^k(\Lambda, \alpha) = \sum_{v \in \alpha(l)} v^{-k}.$$

It will be possible to see from each context whether the weak or the strong modular form E_l^k is intended.

It is easy to see that each E_l^k is homogeneous of degree $-k$.

In the case of $k = 1$, we encounter something familiar:

5.28 Proposition. *$E_l^1(\Lambda, \iota) = \mu_{\Lambda,\iota}(l)$.*

Proof. If $l \in \operatorname{Im} \iota$, let λ' be a representative element of $\iota^{-1}(l)$. Then by Proposition 2.15 we have that

$$E_l^1(\Lambda, \iota) = \sum_{v \in \iota^{-1}(l)} \frac{1}{v} = \sum_{\lambda \in \Lambda} \frac{1}{\lambda' + \lambda} = \frac{1}{e_\Lambda(\lambda')} = \frac{1}{e_\Lambda(\iota^{-1}(l))} = \mu_{\Lambda,\iota}(l).$$

For $l \notin \operatorname{Im} \iota$, both sides are 0, and so the proof is complete. □

5.29 Corollary. *E_l^1 is continuous on $\overleftarrow{\mathcal{L}_N^r}$.*

Proof. By Proposition 5.28 and the appearance of $\mu_{\Lambda,\iota}$ in the metric for $\overleftarrow{\mathcal{L}_N^r}$, if $\mathrm{d}_{\overleftarrow{\mathcal{L}_N^r}}\big((\Lambda_1, \iota_1), (\Lambda_2, \iota_2)\big)$ is small then so is $|E_l^1(\Lambda_1, \iota_1) - E_l^1(\Lambda_2, \iota_2)|$; hence E_l^1 is continuous (in fact, uniformly so). □

There is in fact some dependence between the E_l^k and the modular forms $e_{\Lambda,i}$:

5.30 Proposition. *For $k \in \mathbb{N}$ there is a polynomial $g_k(x, y_1, y_2, \dots)$ with coefficients in $\mathbb{F}_p$ such that $E_l^k = g_k(E_l^1, e_{\Lambda,1}, e_{\Lambda,2}, \dots)$ for all $l \in (N^{-1}/A)^r - \{0\}$.*

Proof. Consider the generating function of the E_l^k. For $l \in \operatorname{Im}\iota$:

$$\begin{aligned}
&\sum_{k>0} E_l^k(\Lambda,\iota) z^k \\
&= \sum_{v\in\iota^{-1}(l)} \sum_{k>0} v^{-k} z^k = \sum_{v\in\iota^{-1}(l)} \frac{z/v}{1-z/v} = z \sum_{v\in\iota^{-1}(l)} \frac{1}{v-z} \\
&= \frac{z}{e_\Lambda(\iota^{-1}(l)-z)} = \frac{z}{e_\Lambda(\iota^{-1}(l)) - e_\Lambda(z)} = \frac{z}{e_\Lambda(\iota^{-1}(l))}\left[1 - \frac{e_\Lambda(z)}{e_\Lambda(\iota^{-1}(l))}\right]^{-1} \\
&= \sum_{n>0} \frac{z e_\Lambda(z)^{n-1}}{e_\Lambda(\iota^{-1}(l))^n} = \sum_{n>0} z e_\Lambda(z)^{n-1} (E_l^1)^n.
\end{aligned}$$

For $l \notin \operatorname{Im}\iota$, both sides are zero, and so the equality still holds. So in general,

$$E_l^k = \left[z^k\right] \sum_{n>0} E_l^n z^n = \left[z^k\right] \sum_{n>0} z e_\Lambda(z)^{n-1} (E_l^1)^n = \sum_{n>0} (E_l^1)^n \left[z^{k-1}\right] e_\Lambda(z)^{n-1}.$$

Now each $\left[z^{k-1}\right] e_\Lambda(z)^{n-1}$ for $n > 0$ is a polynomial in the $e_{\Lambda,i}$ with coefficients in $\mathbb{F}_p$. Moreover, since $e_\Lambda(z)$ has zero constant term, $\left[z^{k-1}\right] e_\Lambda(z)^{n-1} = 0$ for $n > k$, and so the last sum above is finite. This proves the claim. □

5.31 Corollary. *Each E_l^k is continuous on $\overleftarrow{\mathcal{L}_N^r}$.*

Proof. By Corollary 5.29 and Proposition 5.21, E_l^1 and each of the $e_{\Lambda,i}$ are continuous, and so E_l^k, being a polynomial in these, is too. □

5.32 Proposition. *The functions $E_l^k(\Lambda,\iota)$ are holomorphic on $\mathcal{L}_N^r$.*

Proof. We will prove this for the weak Eisenstein series $E_l^k(\Lambda,\alpha)$, as it coincides with the strong Eisenstein series on the main stratum $\mathcal{L}_N^r$. The proof is similar to that of Lemma 5.20.

Recall the isomorphism $\mathcal{L}_N^r \simeq \mathrm{GL}_r(F)\backslash\big((\Omega^r \times \mathbb{C}_\infty^\times) \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)/K(N)\big)$, and consider $E_l^k(\Lambda,\alpha)$ as a function on $((\omega,\psi_r),g) \in (\Omega^r \times \mathbb{C}_\infty^\times) \times \mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$. Since the topology on $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$ is discrete, we need only consider how $E_l^k(\Lambda,\alpha)$ changes as ω and ψ_r vary; hence let g be constant, and define $\Lambda_g = F^r \cap g\hat{A}^r$ and $\Lambda_{g,N} = N^{-1}\Lambda_g = F^r \cap N^{-1}h\hat{A}^r$, strongly discrete 'lattices' in

F^r. Then $\Lambda = \kappa(\omega,\psi_r)\big(F^r \cap g\hat{A}^r\big) = \kappa(\omega,\psi_r)(\Lambda_g)$ with $\kappa(\omega,\psi_r) = \omega \cdot \frac{\psi_r}{\omega_r}$, and letting $a' \in \alpha(l) \subset N^{-1}\Lambda$, there is an $a \in \Lambda_{g,N}$ such that $a' = \kappa(\omega,\psi_r)(a)$; by inspecting the commutative diagram Diagram 3.20 we see that a is independent of ω and ψ_r, depending only on g and l. Thus we have that

$$\alpha(l) = a' + \Lambda = \kappa(\omega,\psi_r)(a) + \kappa(\omega,\psi_r)(\Lambda_g) = \kappa(\omega,\psi_r)(a+\Lambda_g), \quad \text{so that}$$

$$E_l^k(\Lambda,\alpha) = \sum_{\lambda\in\alpha(l)} \lambda^{-k} = \sum_{\lambda\in a+\Lambda_g} \kappa(\omega,\psi_r)(\lambda)^{-k} = \frac{\omega_r^k}{\psi_r^k} \sum_{\lambda\in a+\Lambda_g} (\omega_1\lambda_1+\cdots+\omega_r\lambda_r)^{-k}.$$

Now ψ_r^{-k} is obviously holomorphic as a function of ψ_r, and for each $\lambda \in a+\Lambda_g$, the linear combination $(\omega_1\lambda_1+\cdots+\omega_r\lambda_r)/\omega_r$ is a nonzero holomorphic function of ω, since the ω_i are F_∞-linearly independent. Since $a+\Lambda_g \subset \Lambda_{g,N}$ is strongly discrete and $k > 0$, the above sum converges uniformly on affinoid subsets and hence defines a holomorphic function. □

Hence the E_l^k form our third class of examples of modular forms, and our first class which are not modular forms for $\mathcal{L}^{\leq r}$.

5.33 Proposition. *For $l \in (N^{-1}/A)^r - \{0\}$ and $\gamma \in \mathrm{GL}_r(A/N)$,*

$$E_l^k|\gamma = E_{\gamma^{-1}l}^k.$$

Proof.

$$\begin{aligned}
\big(E_l^k|\gamma\big)(\Lambda,\iota) &= E_l^k(\Lambda,\gamma\iota) \\
&= \begin{cases} \displaystyle\sum_{v\in(\gamma\iota)^{-1}(l)} v^{-k} & \text{if } l \in \mathrm{Im}(\gamma\iota) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \displaystyle\sum_{v\in\iota^{-1}(\gamma^{-1}(l))} v^{-k} & \text{if } \gamma^{-1}(l) \in \mathrm{Im}\,\iota \\ 0 & \text{otherwise} \end{cases} \\
&= E_{\gamma^{-1}l}^k(\Lambda,\iota)
\end{aligned}$$
□

Since these Eisenstein series are our first series of examples which actually depend on the r-inverse level structure ι, it is interesting to see how a partial Eisenstein series looks when restricted to a boundary stratum $\mathcal{L}_N^s$ of $\overleftarrow{\mathcal{L}_N^r}$:

5.34 Proposition. *Let $l \in (N^{-1}/A)^r - \{0\}$, $k \in \mathbb{N}_0$, and δ be an injective selection of* Free_N^r. *Then for each $U \in \mathrm{Free}_N^r$ of rank s, the restriction $E_l^k|_U$ of the* *partial Eisenstein series E_l^k on $\overleftarrow{\mathcal{L}_N^r}$ to the boundary stratum $\mathcal{L}_N^s$ corresponding to U as in Theorem 4.52 is a partial Eisenstein series of rank s, as follows:*

$$E_l^k|_U(\Lambda,\alpha) = \begin{cases} E^k_{\delta_u^{-1}(l)}(\Lambda,\alpha) & l \in \operatorname{Im}\delta_U \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. For $(\Lambda,\alpha) \in \mathcal{L}_N^s$, by Theorem 4.52 we have

$$\begin{aligned} E_l^k|_U(\Lambda,\alpha) &= E_l^k(\Lambda, \delta_U \circ \alpha^{-1}) \\ &= \begin{cases} \displaystyle\sum_{v\in(\delta_U\circ\alpha)^{-1}(l)} v^{-k} & \text{if } l \in \operatorname{Im}(\delta_U\circ\alpha) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \displaystyle\sum_{v\in\alpha^{-1}(\delta_U^{-1}(l))} v^{-k} & \text{if } l \in \operatorname{Im}\delta_U \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} E^k_{\delta_u^{-1}(l)} & \text{if } l \in \operatorname{Im}\delta_U \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

□

Drinfeld module coefficient forms

5.35 Proposition. *For each $a \in A$ and for each ideal I of A, the coefficients $\phi^\Lambda_{a,i} = [\tau^i]\phi^\Lambda_a = [X^{q^i}]\phi^\Lambda_a(X)$ and $\phi^\Lambda_{I,i} = [\tau^i]\phi^\Lambda_I = [X^{q^i}]\phi^\Lambda_I(X)$ for $i \in \mathbb{N}_0$ considered as functions of Λ are modular forms of weight $q^i - 1$ on $\mathcal{L}^{\le r}$.*

Proof. By Proposition 3.1, for each $i \in \mathbb{N}_0$ we have that

$$\begin{aligned} \phi^\Lambda_{a,i} = [X^{q^i}]\phi^\Lambda_a(X) &= a[X^{q^i}]X \sideset{}{'}\prod_{\lambda\in a^{-1}\Lambda/\Lambda}\left(1 - \frac{X}{e_\Lambda(\lambda)}\right) \\ &= a[X^{q^i-1}] \sideset{}{'}\prod_{l\in((a)^{-1}/A)^r}(1 - E_l^1(\Lambda)X) \end{aligned}$$

is a homogeneous symmetric polynomial of degree $q^i - 1$ in the $E_l^1(\Lambda)$ for $l \in ((a)^{-1}/A)^r - \{0\}$, and hence is a modular form of degree $q^i - 1$ on $\overleftarrow{\mathcal{L}^r_{(a)}}$. Moreover, since the aforementioned polynomial is symmetric, $\phi^\Lambda_{a,i}$ is invariant under the action of $\mathrm{GL}_r(A/(a))$ on $\overleftarrow{\mathcal{L}^r_{(a)}}$ and hence is a modular form on $\mathcal{L}^{\le r}$.

The proof for $\phi^\Lambda_{I,i}$ is similar, based instead on Definition 3.4. □

5.36 Corollary. *For each $J \in \mathcal{J}(A)$, $i \in \mathbb{N}_0$, $a \in A$, and ideal I of A, the coefficients $\phi_{a,i}^{J^{-1}\Lambda} = [\tau^i]\phi_a^{J^{-1}\Lambda}$ and $\phi_{I,i}^{J^{-1}\Lambda} = [\tau^i]\phi_I^{J^{-1}\Lambda}$ are modular forms on $\mathcal{L}^{\leq r}$ of weight $q^i - 1$.*

Proof. $\phi_{a,i}^{J^{-1}\Lambda} = \phi_{a,i}^{\Lambda}|J$ and $\phi_{I,i}^{J^{-1}\Lambda} = \phi_{I,i}^{\Lambda}|J$, and so Proposition 5.15 applies. □

We can now prove a proposition which was used in the previous chapter:

4.3 Proposition. *Let $R > 0$ and let Λ and Λ' be lattices of rank $\leq r$, with Λ' being variable. If $\Lambda' \to \Lambda$, i.e.*

$$\mathrm{d}_{\mathcal{L}}(\Lambda', \Lambda) = \sup_{|z|\leq 1} |e_{\Lambda'}(z) - e_\Lambda(z)| \to 0,$$

then

$$\sup_{|z|\leq R} |e_{\Lambda'}(z) - e_\Lambda(z)| \to 0.$$

Proof. Let $a \in A$ such that $|a| \geq R$; since then

$$\sup_{|z|\leq R} |e_{\Lambda'}(z) - e_\Lambda(z)| \leq \sup_{|z|\leq |a|} |e_{\Lambda'}(z) - e_\Lambda(z)|,$$

it is enough to prove the above statement for the case $R = |a|$ for some $a \in A$.

Consider the polynomial ϕ_a^Λ; since each of its finitely many nonzero coefficients is continuous on $\mathcal{L}^{\leq r}$, this polynomial is itself continuous on $\mathcal{L}^{\leq r}$. Thus

$$\begin{aligned}
&\sup_{|z|\leq |a|} |e_{\Lambda'}(z) - e_\Lambda(z)| \\
&= \sup_{|az|\leq |a|} |e_{\Lambda'}(az) - e_\Lambda(az)| \\
&= \sup_{|z|\leq 1} \left|\phi_a^{\Lambda'}(e_{\Lambda'}(z)) - \phi_a^{\Lambda}(e_\Lambda(z))\right| \\
&\leq \sup_{|z|\leq 1} \left|\phi_a^{\Lambda'}(e_{\Lambda'}(z) - e_\Lambda(z))\right| + \sup_{|z|\leq 1} \left|\phi_a^{\Lambda'}(e_\Lambda(z)) - \phi_a^{\Lambda}(e_\Lambda(z))\right|.
\end{aligned}$$

Now since $\Lambda' \to \Lambda$, we have that $\phi_a^{\Lambda'} \to \phi_a^\Lambda$ and so the second term goes to zero since $e_\Lambda(z)$ is bounded on $|z| \leq 1$. For the first term, note that since $\phi_a^{\Lambda'} \to \phi_a^\Lambda$, the coefficients of the polynomial $\phi_a^{\Lambda'}$ are bounded and so since $e_{\Lambda'}(z) - e_\Lambda(z) \to 0$ uniformly on $|z| \leq 1$, the first term also goes to zero. Thus as desired,

$$\sup_{|z|\leq |a|} |e_{\Lambda'}(z) - e_\Lambda(z)| \to 0.$$

□

5.37 There is some dependence between these Drinfeld module coefficient forms for different values of $a \in A$ and ideals I. Indeed, from $\phi^\Lambda_{ab} = \phi^\Lambda_a \circ \phi^\Lambda_b$ we see that each coefficient $\phi^\Lambda_{ab,i}$ can be written as a polynomial in the coefficients $\phi^\Lambda_{a,j}$ and $\phi^\Lambda_{b,j}$ and similarly $\phi^\Lambda_{IJ,i}$ can be written as a polynomial in the $\phi^\Lambda_{I,j}$ and the $\phi^\Lambda_{J,j}$ for ideals I and J. Of course, since each ϕ^Λ_a is a polynomial of degree at most $r \cdot \deg a$ in τ, and exactly that much if Λ has rank equal to r, $\phi^\Lambda_{a,i}$ is zero for $i > r \deg a$.

5.38 Let us also consider the modular forms $\Delta_a(\Lambda) = \phi^\Lambda_{a,r\deg a}$ on $\mathcal{L}^{\le r}$ for $a \in A$ and $\Delta_I(\Lambda) = \phi^\Lambda_{I,r\deg I}$, which are nonzero on the main stratum $\mathcal{L}^r$ since ϕ^Λ_a and ϕ^Λ_I have degree exactly $r \deg a$ and $r \deg I$ respectively. For the same reason, these modular forms are cusp forms, being zero on the boundary $\mathcal{L}^{\le r-1} = \mathcal{L}^{\le r} - \mathcal{L}^r$.

5.39 From the relation $\phi^\Lambda_{ab} = \phi^\Lambda_a \circ \phi^\Lambda_b$ we have that

$$\begin{aligned}
\Delta_{ab}(\Lambda) &= \left[\tau^{r\deg ab}\right]\phi^\Lambda_{ab} = \left[\tau^{r\deg ab}\right]\phi^\Lambda_a\phi^\Lambda_b \\
&= \left[\tau^{r\deg ab}\right]\left(\Delta_a(\Lambda)\tau^{r\deg a} + o\left(\tau^{r\deg a}\right)\right)\left(\Delta_b(\Lambda)\tau^{r\deg b} + o\left(\tau^{r\deg b}\right)\right) \\
&= \left[\tau^{r\deg a + r\deg b}\right]\Delta_a(\Lambda)\tau^{r\deg a}\Delta_b(\Lambda)\tau^{r\deg b} \\
&= \Delta_a(\Lambda)\Delta_b(\Lambda)^{q^{r\deg a}} = \Delta_a(\Lambda)\Delta_b(\Lambda)^{|a|^r}.
\end{aligned}$$

Since $ab = ba$, this implies that $\Delta_a(\Lambda)\Delta_b(\Lambda)^{|a|^r} = \Delta_b(\Lambda)\Delta_a(\Lambda)^{|b|^r}$, so that $\Delta_a(\Lambda)^{|b|^r-1} = \Delta_b(\Lambda)^{|a|^r-1}$. Thus if $\sqrt[|a|^r-1]{\Delta_a(\Lambda)}$ exists in some sense, it would be largely independent of a; the same applies for the ideal-based Drinfeld module coefficients.

Relation with BBP definitions

We will investigate the relation between our modular forms and that defined by Basson, Breuer, and Pink[‡] in [BBP1]. There are some differences in notation between our work and that of [BBP1; BBP2; BBP3]; for instance, there the elements of Ω^r and Ψ^r are considered as column vectors as opposed to row vectors, which results in differences of definition of various actions. For this reason, we will translate relevant definitions and results from [BBP1; BBP2; BBP3] to our context before referring to them.

Let $\xi \in \mathbb{C}_\infty^\times$ be fixed for the remainder of this thesis. Note that each $\omega \in \Omega^r \simeq \Psi^r/\mathbb{C}_\infty^\times$ has a representative $\overline{\omega} \in \Psi^r$ with last component $\overline{\omega}_r = \xi$;

[‡]For the remainder of this chapter, we will largely refer to the authors Basson, Breuer, and Pink collectively as *BBP*.

we denote this representative by $\psi(\omega)$. The map $\Omega^r \to \Psi^r$, $\omega \mapsto \psi(\omega)$ is rigid analytic, since the rigid analytic structure on Ψ^r has been defined to be that of the product $\Omega^r \times \mathbb{C}_\infty$ via the isomorphism Proposition 3.16.

5.40 Definition. For $\gamma \in \mathrm{GL}_r(F)$, $\psi \in \Psi^r$, and $\omega \in \Omega^r$, we define

$$j(\gamma, \psi) := \xi^{-1} \cdot \left(\psi\gamma^{-1}\right)_r,$$

where $(\psi\gamma^{-1})_r$ denotes the last entry of the vector $\psi\gamma^{-1}$, and similarly

$$j(\gamma, \omega) := \xi^{-1} \cdot \left(\psi(\omega)\gamma^{-1}\right)_r = j(\gamma, \psi(\omega)).$$

This $j(\gamma, \psi)$ serves as a normalisation factor, preserving the last component being equal to ξ under the action of $\mathrm{GL}_r(F)$:

5.41 Proposition. *For $\omega \in \Omega^r$ and $\gamma \in \mathrm{GL}_r(F)$,*

$$\psi(\omega\gamma^{-1}) = j(\gamma, \psi(\omega))^{-1} \cdot \psi(\omega)\gamma^{-1} = j(\gamma, \omega)^{-1} \cdot \psi(\omega)\gamma^{-1}.$$

Proof. With $\simeq$ temporarily denoting similarity up to a multiple of $\mathbb{C}_\infty^\times$, $\psi(\omega\gamma^{-1}) \simeq \omega\gamma^{-1} \simeq \psi(\omega)\gamma^{-1}$. Inspecting the last components of the first and last terms yields the relevant scaling factor $j(\gamma, \psi(\omega))$. $\square$

5.42 Definition. For $f : \Omega^r \to \mathbb{C}_\infty$, $k \in \mathbb{Z}$, and $\gamma \in \mathrm{GL}_r(F)$, we define

$$f|_k\gamma : \Omega^r \to \mathbb{C}_\infty, \qquad (f|_k\gamma)(\omega) := j(\gamma, \omega)^{-k} f(\omega\gamma^{-1}).$$

It is easily shown that this 'slash operator' $|_k$ induces a right action of $\mathrm{GL}_r(F)$ on the set of functions $\Omega^r \to \mathbb{C}_\infty$.

For future reference, we include here the definition of a weak modular form in [BBP1], which we will refer to as a *weak BBP modular form* to contrast with the weak modular forms defined earlier.[§]

5.43 Definition. Consider an integer k and an arithmetic subgroup $\Gamma < \mathrm{GL}_r(F)$. A *weak BBP modular form f of weight k for* Γ is a holomorphic function $f : \Omega^r \to \mathbb{C}_\infty$ such that for all $\gamma \in \Gamma$, $f|_k\gamma = f$.

The $\mathbb{C}_\infty$-vector space of weak BBP modular forms of weight k for Γ will be denoted by $\mathcal{W}_k(\Gamma)$, and the graded $\mathbb{C}_\infty$-algebra of weak BBP modular forms for Γ will be denoted by $\mathcal{W}_*(\Gamma)$.

[§] We do not include the 'type' parameter m investigated in BBP's work; all the modular forms considered in this thesis are of type 0.

With the decomposition of $\mathcal{L}_N^r$ into $\bigsqcup_{g\in H} \Gamma_g\backslash\Psi^r$ in Proposition 3.25 in mind, with H being a set of representatives in $\mathrm{GL}_r(\mathbb{A}_F^{fin})$ for the double quotient $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)$ and $\Gamma_g = gK(N)g^{-1}\cap\mathrm{GL}_r(F)$ for each $g\in H$, we define maps from modular forms on $\mathcal{L}_N^r$ to BBP modular forms on the quotients $\Gamma_g\backslash\Omega^r$. These maps essentially separate the function f into its values on each of the irreducible components of $\mathcal{L}_N^r$.

5.44 Definition. For a function $f:\mathcal{L}_N^r\to\mathbb{C}_\infty$ and $g\in\mathrm{GL}_r(\mathbb{A}_F^{fin})$, we define the function $f_g:\Omega^r\to\mathbb{C}_\infty$ by

$$f_g(\omega) = f(\Theta([\psi(\omega),g]));$$

here Θ is the bijection defined in Theorem 3.19.

For $f:\overleftarrow{\mathcal{L}_N^r}\to\mathbb{C}_\infty$, we let f_g denote the same construction using the restriction of f to the main stratum $\mathcal{L}_N^r$ of $\overleftarrow{\mathcal{L}_N^r}$.

5.45 Proposition. *For $f\in\mathbf{Weak}_N^{k,r}$ and $\gamma\in\mathrm{GL}_r(F)$, $f_g|_k\gamma = f_{\gamma^{-1}g}$.*

Proof. Let $\omega\in\Omega^r$. Then since f is homogeneous of degree $-k$,

$$\begin{aligned}(f_g|_k\gamma)(\omega) &= j(\gamma,\omega)^{-k}f_g(\omega\gamma^{-1})\\ &= j(\gamma,\omega)^{-k}f\big(\Theta([\psi(\omega\gamma^{-1}),g])\big)\\ &= j(\gamma,\omega)^{-k}f\big(\Theta([j(\gamma,\psi(\omega))^{-1}\cdot\psi(\omega)\gamma^{-1},g])\big)\\ &= j(\gamma,\omega)^{-k}f\big(j(\gamma,\omega)^{-1}\Theta([\psi(\omega)\gamma^{-1},g])\big)\\ &= j(\gamma,\omega)^{-k}j(\gamma,\omega)^{k}f\big(\Theta([\psi(\omega),\gamma^{-1}g])\big)\\ &= f_{\gamma^{-1}g}(\omega)\end{aligned}$$

since $[\psi\gamma^{-1},g] = [\psi,\gamma^{-1}g]$ because $\gamma\in\mathrm{GL}_r(F)$. $\square$

5.46 Theorem. *If $f\in\mathbf{Weak}_N^{k,r}$ is a weak modular form of weight k, then each f_g is a weak BBP modular form for Γ_g.*

Proof. Since f, Θ, and the map $\omega\mapsto\psi(\omega)$ are rigid analytic, f_g is also rigid analytic. So it remains to show that f_g satisfies the relevant transformation relation in Definition 5.43. Now $f_g|_k\gamma = f_{\gamma^{-1}g}$ and $g^{-1}\gamma g\in K(N)$, so that in the quotient $\mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)$ we have $[\gamma^{-1}g] = [\gamma^{-1}gg^{-1}\gamma g] = [g]$. Thus

$$f_{\gamma^{-1}g}(\omega) = f\big(\Theta([\psi(\omega),\gamma^{-1}g])\big) = f(\Theta([\psi(\omega),g])) = f_g(\omega)$$

as desired. $\square$

5.47 Proposition. *For each $g \in \mathrm{GL}_r(\mathbb{A}_F^{fin})$ the map*

$$\mathbf{Weak}_N^r \to \mathcal{W}_*(\Gamma_g), \qquad f \mapsto f_g$$

is a homomorphism of graded $\mathbb{C}_\infty$-algebras.

Proof. Theorem 5.46 shows that the above map sends a weak modular form of weight k to a weak BBP modular form of weight k, and from Definition 5.44 it preserves scaling by $\mathbb{C}_\infty$ and multiplication of weak modular forms. □

5.48 Proposition. *If H is a set of representatives in $\mathrm{GL}_r(\mathbb{A}_F^{fin})$ for the double quotient $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)$, then the following homomorphism of graded $\mathbb{C}_\infty$-algebras is injective:*

$$\mathbf{Weak}_N^r \to \prod_{g\in H} \mathcal{W}_*(\Gamma_g), \qquad f \mapsto (f_g)_{g\in H}.$$

Proof. By Proposition 5.47, it is enough to show that if if $f \in \mathbf{Weak}_N^{k,r}$ is mapped to $0 = (0)_{g\in H}$ then $f = 0$. So assume f to be such that $f_g = 0$ for all $g \in H$.

So let $g \in H$, so that $f_g = f(\Theta([\psi(\omega), g])) = 0$ for all $\omega \in \Omega^r$. Then scaling by $\mathbb{C}_\infty^\times$ we see that $f(\Lambda, \alpha) = 0$ for all (Λ, α) in the irreducible component corresponding to g. Since H is a complete set of representatives for $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)$, $f = 0$ over all the irreducible components of $\mathcal{L}_N^r$ and thus is identically zero. □

Now after our definition of weak modular forms two sections ago we defined strong modular forms, which can be seen as weak modular forms which satisfy continuity conditions at the boundary ∂_N^r of $\overleftarrow{\mathcal{L}_N^r}$. Similarly, BBP modular forms are defined as weak BBP modular forms which satisfy an additional condition 'at infinity':

Holomorphicity at infinity and *going to zero at infinity* are conditions regarding the behaviour of a holomorphic function on Ω^r as the first component of $\omega \in \Omega^r$ goes to infinity, which we will not repeat here. We will, however, include the following characterisation of holomorphicity at infinity and going to zero at infinity which was communicated to us by BBP; the proof is included in Appendix 1:

5.49 Proposition. *Let $\Gamma < \mathrm{GL}_r(F)$ be an arithmetic subgroup and $f : \Omega^r \to \mathbb{C}_\infty$ be a holomorphic function such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$.*

Then f is holomorphic at infinity if and only if it is bounded on every vertical line, i.e. for every vector $\psi' = (\psi_2, \ldots, \psi_r) \in \Psi^{r-1}$ there are real numbers $N > 0$ and $R > 0$ such that for all $\psi_1 \in \mathbb{C}_\infty$ satisfying $\mathrm{d}(\psi_1, \psi' F_\infty^{r-1}) > R$ we have $|f(\psi_1 : \ldots : \psi_r)| < N$.

Moreover, f goes to zero at infinity if and only if it goes to zero on each vertical line, i.e. for every $\psi' = (\psi_2, \ldots, \psi_r) \in \Psi^{r-1}$ and $\epsilon > 0$ there is an $R > 0$ such that $|f(\psi_1 : \ldots : \psi_r)| < \epsilon$ for $\mathrm{d}(\psi_1, \psi' F_\infty^{r-1}) > R$.

5.50 Definition. For an integer k and an arithmetic subgroup $\Gamma < \mathrm{GL}_r(F)$, a *strong BBP modular form f of weight k for Γ* is a weak BBP modular form such that $f|_k\gamma$ is holomorphic at infinity for all $\gamma \in \mathrm{GL}_r(F)$.

The $\mathbb{C}_\infty$-vector space of strong BBP modular forms of weight k for Γ will be denoted by $\mathcal{M}_k(\Gamma)$, with the graded $\mathbb{C}_\infty$-algebra of all strong BBP modular forms denoted by $\mathcal{M}_*(\Gamma)$. Also, the adjective *strong* will be omitted unless contrasting with BBP *weak* modular forms.

5.51 Definition. A *BBP cusp form of weight k for Γ* is a strong BBP modular form f such that $f|_k\gamma$ goes to zero at infinity for all $\gamma \in \mathrm{GL}_r(F)$

The $\mathbb{C}_\infty$-vector space of BBP cusp forms of weight k for Γ will be denoted by $\mathcal{S}_k(\Gamma)$, with the graded $\mathbb{C}_\infty$-algebra of all BBP cusp forms denoted by $\mathcal{S}_*(\Gamma)$.

Using Proposition 5.49, we can prove the following:

5.52 Proposition. *If $f \in \mathbf{Strong}_N^{k,r}$ is a strong modular form and $g \in \mathrm{GL}_r\left(\mathbb{A}_F^{fin}\right)$, then f_g is a strong BBP modular form of weight k for Γ_g.*

Proof. Let $\gamma \in \mathrm{GL}_r(F)$ and $\psi' \in \Psi^{r-1}$, where without loss of generality $\psi_r = \xi$, and let $\psi = (\psi_1, \ldots, \psi_r)$ where $\psi_1 \in \mathbb{C}_\infty$ is variable. Also let $F^r \cap \gamma^{-1} g\hat{A}^r = (I_1, I_2, \ldots, I_r)^T$ where the I_i are fractional ideals, so that for $(\Lambda, \alpha) = \Theta([\psi, \gamma^{-1}g])$ we have $\Lambda = \psi(F^r \cap \gamma^{-1}g\hat{A}^r) = I_1\psi_1 + I_2\psi_2 + \cdots I_r\psi_r$. Then as in Proposition 4.7 we have that as $\mathrm{d}(\psi_1, \psi' F_\infty^{r-1}) \to \infty$ the lattice Λ tends to $\Lambda' := I_2\psi_2 + \cdots + I_r\psi_r$. Also, as in the proof of Proposition 4.31 the r-inverse level N structure $\iota = \alpha^{-1}$ tends to the r-inverse level N structure

$$\iota' : N^{-1}\Lambda'/\Lambda' \hookrightarrow \left(N^{-1}/A\right)^r, \quad [\lambda_2\psi_2 + \cdots + \lambda_r\psi_r]_{\Lambda'} \mapsto \iota([\lambda_2\psi_2 + \cdots + \lambda_r\psi_r]_\Lambda).$$

Thus as $\mathrm{d}(\psi_1, \psi' F_\infty^{r-1}) \to \infty$ we have that $(\Lambda, \iota) \to (\Lambda', \iota')$, with (Λ', ι') not dependent on ψ_1. Thus since f is continuous on $\overleftarrow{\mathcal{L}_N^r}$ we have that

$$(f_g|_k\gamma)(\omega) = f_{\gamma^{-1}g}(\omega) = f(\Lambda, \iota) \to f(\Lambda', \iota');$$

since the limit exists, $f_g|_k\gamma$ is bounded on the vertical line defined by ψ'. □

5.53 Proposition. *If $f \in \mathbf{Cusp}_N^{k,r}$ is a cusp form and $g \in \mathrm{GL}_r(\mathbb{A}_F^{fin})$, then f_g is a BBP cusp form of weight k for Γ_g.*

Proof. Let $\gamma \in \mathrm{GL}_r(F)$. Then similarly to Proposition 5.52,

$$(f_g|_k\gamma)(\omega) = f_{\gamma^{-1}g}(\omega) = f(\Lambda, \iota) \to f(\Lambda', \iota') = 0,$$

the zero due to f being a cusp form. Since the limit is zero, $f_g|_k\gamma$ goes to zero at infinity on the vertical line defined by ψ'. □

5.54 Theorem. *If H is a set of representatives in $\mathrm{GL}_r(\mathbb{A}_F^{fin})$ for the double quotient $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}_F^{fin})/K(N)$, then the following homomorphisms of graded $\mathbb{C}_\infty$-algebras are injective:*

$$\mathbf{Strong}_N^r \to \prod_{g\in H} \mathcal{M}_*(\Gamma_g), \qquad \mathbf{Cusp}_N^r \to \prod_{g\in H} \mathcal{S}_*(\Gamma_g), \qquad f \mapsto (f_g)_{g\in H}.$$

Proof. These are shown by Propositions 5.48, 5.52 and 5.53. □

5.55 Corollary. *For each $k \in \mathbb{Z}$, $\mathbf{Strong}_N^{k,r}$ has finite dimension as a $\mathbb{C}_\infty$-vector space.*

Proof. By Proposition 5.52 and Theorem 5.54, $\mathbf{Strong}_N^{k,r}$ can be seen as a subspace of $\prod_{g\in H} \mathcal{M}_k(\Gamma_g)$; by [BBP2, Theorem 11.1] the latter space is finite dimensional. □

5.56 The above injection of strong modular forms is not a bijection in general, as by Theorem 4.38 the irreducible components of $\mathcal{L}_N^r$ share a common boundary and hence the condition of continuity on $\overleftarrow{\mathcal{L}_N^r}$ imposes relations between the modular form's value on different irreducible components, whereas in the product $\prod_{g\in H} \mathcal{M}_*(\Gamma_g)$ the modular forms on each component are completely independent. As a more explicit example, consider the modular form of weight 0 on $\prod_{g\in H} \mathcal{M}_*(\Gamma_g)$ which is defined to have constant value 1 on $\mathcal{M}_*(\Gamma_{g_0})$ for some particular $g_0 \in \mathrm{GL}_r(\mathbb{A}_F^{fin})$ and constant value 0 on $\mathcal{M}_*(\Gamma_g)$ for $g \in H - \{g_0\}$. However, the above injection does become a bijection when restricting to cusp forms, which are zero on the boundary:

5.57 Theorem. *The map $f \mapsto (f_g)_{g\in H}$ is a bijection between the spaces $\mathbf{Cusp}_N^r$ and $\prod_{g\in H} \mathcal{S}_*(\Gamma_g)$.*

Proof. Let $(f_g)_{g\in H} \in \prod_{g\in H} \mathcal{S}_k(\Gamma_g)$ be a tuple of BBP cusp forms of weight k, and define the weak modular form $f \in \mathbf{Cusp}_N^{k,r}$ by $f(t\Theta([\psi(\omega), g])) = t^{-k} f_g(\omega)$ for $\omega \in \Omega^r$, $t \in \mathbb{C}_\infty$ and $g \in H$; the holomorphicity and homogeneity of f follows from the corresponding properties of the f_g. All that remains is to show that f is continuous when considered as a function on $\overleftarrow{\mathcal{L}^r_N}$, defined to be zero on the boundary $\partial^r_N = \overleftarrow{\mathcal{L}^r_N} - \mathcal{L}^r_N$. However, since by Corollary 4.37 the unions $C_g \cup \partial^r_N$, where C_g denotes the irreducible component of $\mathcal{L}^r_N$ corresponding to $g \in H$, are each closed in $\overleftarrow{\mathcal{L}^r_N}$ and together cover the whole space $\overleftarrow{\mathcal{L}^r_N}$, and there are finitely many such, it is enough to show that f is continuous on each $C_g \cup \partial^r_N$; as in the proofs of Propositions 5.52 and 5.53, this follows from the fact that each f_g is a cusp form. $\square$

6 Conclusion

In this thesis we have presented a theory of modular forms of arbitrary finite rank r, similarly to the work of Gekeler and Basson, Breuer, and Pink. In contrast to these other works, ours does not make much use of rigid analysis and uses continuity in a metric space to define when a weak modular form is in fact a strong modular form. Hence it may be more accessible to those unfamiliar with rigid analysis. We have also introduced actions of $\mathrm{GL}_r(A/N)$ and $\mathcal{J}(A)$ on these modular forms, which as far as we can tell is novel and will hopefully have effects on the general theory of modular forms.

There are a number of directions in which this work may be extended, which we hope will be investigated in future:

- Dimension formulae for the vector spaces of modular forms of weight k.
- The generation of the graded ring of modular forms by modular forms of small weight. For instance, in the case of $A = \mathbb{F}_q[T]$ and rank 2 it has been shown that the ring of BBP modular forms for the full congruence subgroup $\Gamma(N) = \ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(A/N)) = K(N) \cap \mathrm{GL}_2(F)$ is generated by weight 1 Eisenstein series and possibly some weight 2 cusp forms (see [Cor97b] for more details); can this be extended to modular forms on $\overleftarrow{\mathcal{L}_N^r}$?
- A characterisation of when the action of the group $\mathcal{J}(A)$ on $\mathcal{L}_N^r$ extends to a homeomorphism on $\overleftarrow{\mathcal{L}_N^r}$ and thus to an action on modular forms for $K(N)$.
- Generalising the principal congruence subgroup $K(N)$ to a general compact open subgroup $\mathcal{K}$ of $\mathrm{GL}_r\big(\mathbb{A}_F^{fin}\big)$.
- Including the type $m \in \mathbb{Z}$ of a modular form as in [BBP1; BBP2; BBP3].
- Establishing a notion of successive minimum basis (SMB) of a lattice and fundamental domain $\mathbf{F}$ of Ω^r for general $A \neq \mathbb{F}_q[T]$, the $\mathbb{F}_q[T]$ case handled by [Gek19, Section 3].

Bibliography

[Bas14] Dirk Basson. 'On the coefficients of Drinfeld modular forms of higher rank'. PhD thesis. Stellenbosch University, 2014. HDL: `10019.1/86387` (cit. on p. viii).

[Bas17] Dirk Basson. 'A product formula for the higher rank Drinfeld discriminant function'. In: *Journal of Number Theory* 178 (2017), pp. 190–200. DOI: `10.1016/j.jnt.2017.02.010` (cit. on p. viii).

[BB17] Dirk Basson and Florian Breuer. 'On certain Drinfeld modular forms of higher rank'. In: *Journal de Théorie des Nombres de Bordeaux* 29.3 (2017), pp. 827–843. JSTOR: `26274100` (cit. on p. viii).

[BBP1] Dirk Basson, Florian Breuer and Richard Pink. 'Drinfeld modular forms of arbitrary rank, Part I: Analytic Theory'. In: (2018). arXiv: `1805.12335 [math.NT]` (cit. on pp. viii, ix, 7, 72, 73, 79, 84–87).

[BBP2] Dirk Basson, Florian Breuer and Richard Pink. 'Drinfeld modular forms of arbitrary rank, Part II: Comparison with Algebraic Theory'. In: (2018). arXiv: `1805.12337 [math.NT]` (cit. on pp. viii, ix, 72, 77, 79, 84).

[BBP3] Dirk Basson, Florian Breuer and Richard Pink. 'Drinfeld modular forms of arbitrary rank, Part III: Examples'. In: (2018). arXiv: `1805.12339 [math.NT]` (cit. on pp. viii, ix, 63, 72, 79, 84).

[BR09] Florian Breuer and Hans-Georg Rück. 'Drinfeld modular polynomials in higher rank'. In: *Journal of Number Theory* 129.1 (2009), pp. 59–83. DOI: `10.1016/j.jnt.2008.07.010` (cit. on p. viii).

[Bre10] Florian Breuer. 'Torsion bounds for elliptic curves and Drinfeld modules'. In: *Journal of Number Theory* 130.5 (2010), pp. 1241–1250 (cit. on p. 55).

[Cor97a] Gunther Cornelissen. 'A survey of Drinfeld modular forms'. In: *Proceedings of the workshop on Drinfeld modules, modular schemes and applications*. Ed. by Ernst-Ulrich Gekeler et al. World Scientific. 1997, pp. 167–187 (cit. on p. viii).

[Cor97b] Gunther Cornelissen. 'Drinfeld modular forms of weight one'. In: *Journal of Number Theory* 67.2 (1997), pp. 215–228. ISSN: 0022-314X. DOI: 10.1006/jnth.1997.2185 (cit. on p. 79).

[Dri|en] V.G. Drinfel'd. 'Elliptic modules'. English. In: *Mathematics of the USSR-Sbornik* 23.4 (Apr. 1974). Translation of [Dri|ru], pp. 561–592. DOI: 10.1070/sm1974v023n04abeh001731 (cit. on pp. viii, 13, 14, 17).

[Dri|ru] В. Г. Дринфельд. 'Эллиптические модули'. Russian. In: *Matematicheskiĭ Sbornik. Novaya Seriya* 94(136).4(8) (1974), pp. 594–627. URL: http://mi.mathnet.ru/msb3735 (cit. on p. 81).

[FP|RAG] Jean Fresnel and Marius Van der Put. *Rigid Analytic Geometry and Its Applications*. Vol. 218. Progress in Mathematics. Original French edition published in 1981. Birkhäuser Basel, 2012. 299 pp. ISBN: 978-0-8176-4206-8. DOI: 10.1007/978-1-4612-0041-3 (cit. on p. 33).

[Ge|DMC] Ernst-Ulrich Gekeler. *Drinfeld Modular Curves*. Vol. 1231. Lecture Notes in Mathematics. Springer, 2006. 108 pp. DOI: 10.1007/BFb0072692 (cit. on pp. viii, 7).

[Gek19] Ernst-Ulrich Gekeler. 'Towers of GL(r)-type of modular curves'. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2019.754 (2019), pp. 87–141. DOI: 10.1515/crelle-2017-0012 (cit. on p. 79).

[Gek86] Ernst-Ulrich Gekeler. 'Satake compactification of Drinfeld modular schemes'. In: *Proceedings of the Conference on p-adic Analysis, Hengelhoef*. Ed. by L van Hamme and N de Grande-De Kimpe. Vrije Universiteit Brussel, 1986.

[Gek89] Ernst-Ulrich Gekeler. 'Quasi-periodic functions and Drinfeld modular forms'. In: *Compositio Mathematica* 69.3 (1989), pp. 277–293. URL: http://www.numdam.org/item/?id%20=%20CM_1989__69_3_277_0.

[Gek99] Ernst-Ulrich Gekeler. 'A survey on Drinfeld modular forms'. In: *Turkish J. Math* 23.4 (1999), pp. 485–518 (cit. on p. viii).

[GHR|1] Ernst-Ulrich Gekeler. 'On Drinfeld modular forms of higher rank'. In: *Journal de Théorie des Nombres de Bordeaux* 29.3 (2017), pp. 875–902. DOI: 10.5802/jtnb.1005 (cit. on pp. viii, ix).

[GHR|2] Ernst-Ulrich Gekeler. 'On Drinfeld modular forms of higher rank II'. In: *Journal of Number Theory* (2019). ISSN: 0022-314X. DOI: 10.1016/j.jnt.2018.11.011 (cit. on pp. viii, ix).

[GHR|3] Ernst-Ulrich Gekeler. 'On Drinfeld modular forms of higher rank III: The analogue of the $k/12$-formula'. In: *Journal of Number Theory* 192 (2018), pp. 293–306. ISSN: 0022-314X. DOI: 10.1016/j.jnt.2018.04.018 (cit. on pp. viii, ix).

[GHR|4] Ernst-Ulrich Gekeler. 'On Drinfeld modular forms of higher rank IV: Modular forms with level'. In: *Journal of Number Theory* (2019). ISSN: 0022-314X. DOI: 10.1016/j.jnt.2019.04.019 (cit. on pp. viii, ix).

[Go|Bas] David Goss. *Basic Structures of Function Field Arithmetic*. Springer Science & Business Media, 2012. 424 pp. ISBN: 978-3-540-63541-3. DOI: 10.1007/978-3-642-61480-4 (cit. on pp. 4, 5, 7, 11, 33).

[Gos80] David Goss. 'π-adic Eisenstein series for function fields'. In: *Compositio Mathematica* 41.1 (1980), pp. 3–38. URL: http://www.numdam.org/item/CM_1980__41_1_3_0 (cit. on p. viii).

[Gos92] David Goss. 'Some integrals attached to modular forms in the theory of function fields'. In: *The Arithmetic of Function Fields*. Ed. by David Goss, Michael I Rosen and David R Hayes. Ohio State University Mathematical Research Institute publications. Walter de Gruyter, 1992, pp. 227–251 (cit. on p. viii).

[Ham02] Yoshinori Hamahata. 'On a product expansion for the Drinfeld discriminant function'. In: *Journal of the Ramanujan Mathematical Society* 17.3 (2002), pp. 1–13.

[Hub13] Patrik Hubschmid. 'The André–Oort conjecture for Drinfeld modular varieties'. In: *Compositio Mathematica* 149.4 (2013), pp. 507–567. DOI: 10.1112/S0010437X12000681 (cit. on pp. 17, 18).

[Kap|en] Mikhail Mikhailovich Kapranov. 'On cuspidal divisors on the modular varieties of elliptic modules'. English. In: *Mathematics of the USSR-Izvestiya* 30.3 (June 1988). Translation of [Kap|ru], pp. 533–547. DOI: 10.1070/im1988v030n03abeh001029 (cit. on p. viii).

[Kap|ru] Михаил Михайлович Капранов. 'О параболических дивизорах на модулярных многообразиях эллиптических модулей'. In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 51.3 (1987), pp. 568–583. URL: http://mi.mathnet.ru/izv1309 (cit. on p. 82).

[Kat73] Nicholas M Katz. 'P-adic properties of modular schemes and modular forms'. In: *Modular functions of one variable III*. Ed. by Willem Kuijk and Jean-Pierre Serre. Vol. 350. Lecture Notes in Mathematics. Springer, 1973, pp. 69–190. ISBN: 978-3-540-06483-1. DOI: `10.1007/978-3-540-37802-0_3` (cit. on p. viii).

[Ne|ANT] Jürgen Neukirch. *Algebraic Number Theory*. Trans. German by N. Schappacher. Vol. 322. Grundlehren der mathematischen Wissenschaften. Springer Science & Business Media, 2013. 574 pp. ISBN: 978-3-540-65399-8. DOI: `10.1007/978-3-662-03983-0` (cit. on p. 36).

[Per14] Rudolph Perkins. 'The Legendre determinant form for Drinfeld modules in arbitrary rank'. In: (2014). arXiv: `1409.6693 [math.NT]`.

[Pin13] Richard Pink. 'Compactification of Drinfeld modular varieties and Drinfeld modular forms of arbitrary rank'. In: *Manuscripta mathematica* 140.3-4 (2013), pp. 333–361. DOI: `10.1007/s00229-012-0544-3` (cit. on pp. viii, 13, 14).

[Pin19] Richard Pink. 'Compactification of Drinfeld Moduli Spaces as Moduli Spaces of A-Reciprocal Maps and Consequences for Drinfeld Modular Forms'. In: *arXiv preprint arXiv:1903.02432* (2019). arXiv: `1903.02432 [math.NT]` (cit. on p. viii).

[SS91] Peter Schneider and Ulrich Stuhler. 'The cohomology of p-adic symmetric spaces'. In: *Inventiones Mathematicae* 105.1 (1991), pp. 47–122. DOI: `10.1007/BF01232257` (cit. on pp. 14, 17).

[St|FFC] Henning Stichtenoth. *Algebraic Function Fields and Codes*. 2nd ed. Vol. 254. Graduate Texts in Mathematics. Springer Science & Business Media, 2009. 360 pp. ISBN: 978-3-540-76877-7. DOI: `10.1007/978-3-540-76878-4`.

[Wof|SE] Eric Wofsey. *Extension of linearly independent set to a basis of a free module over a quotient of a Dedekind domain*. 20th Aug. 2019. URL: `https://math.stackexchange.com/a/3329375/108806` (cit. on p. 48).

Appendices

Vertical lines in BBP

The results in this section will largely concern a selection of definitions and properties defined in [BBP1; BBP2; BBP3], all of which will simply be restated from these papers, except for Definition .4 and Theorem .5. We will only concern ourselves with the most relevant definitions and propositions necessary for the statement and proof of Theorem .5.

For this section, let Γ be a fixed arithmetic subgroup of $\mathrm{GL}_r(F)$. We also let U denote the algebraic subgroup of $\mathrm{GL}_r(F)$ of matrices of the form

$$\begin{pmatrix} 1 & 0 \cdots 0 \\ * & \\ \vdots & \mathrm{Id}_{r-1} \\ * & \end{pmatrix}$$

where Id_{r-1} denotes the identity matrix of size $(r-1) \times (r-1)$, and denote $\Gamma_U = \Gamma \cap U$, with the 'lattice' $L' \subset F^{r-1}$ of elements where the corresponding element of U lies in Γ_U.

Also, for every $\omega' \in \Omega^{r-1}$ and $\omega_1 \in \mathbb{C}_\infty$, we will let $\omega = (\omega_1, \omega') \in \Omega^r$; here ω' and ω are normalised so that the last component is equal to ξ. Also, $\omega' L'$ is a lattice of rank $r-1$ which does not include ω_1; thus

$$u_{\omega'}(\omega_1) \coloneqq \frac{1}{e_{\omega' L'}(\omega_1)} \in \mathbb{C}_\infty$$

is well-defined for all $\omega \in \Omega^r$.

.1 Definition. For a function $f : \Omega^r \to \mathbb{C}_\infty$ and a subgroup Γ_U of $\mathrm{GL}_r(F)$, we say that f is Γ_U-invariant if and only if $f(\omega\gamma) = f(\omega)$ for all $\gamma \in \Gamma_U$ and $\omega \in \Omega^r$.

Any modular form for Ω^r is Γ_U-invariant, since $\det \gamma = 1$ for $\gamma \in \Gamma_U$.

The following proposition is an essential part of BBP's work:

.2 Proposition. *([BBP1, Proposition 5.4]) For any Γ_U-invariant holomorphic function $f : \Omega^r \to \mathbb{C}_\infty$ there exist unique holomorphic functions $f_k : \Omega^{r-1} \to \mathbb{C}_\infty$ for $k \in \mathbb{Z}$ such that the series*

$$\sum_{k\in\mathbb{Z}} f_k(\omega') \cdot u_{\omega'}(\omega_1)^k$$

converges to $f(\omega_1, \omega')$ on some neighbourhood of infinity, and uniformly on every affinoid subset thereof.

.3 Definition. ([BBP1, Definition 5.12]) A Γ_U-invariant holomorphic function $f : \Omega^r \to \mathbb{C}_\infty$ is said to be *holomorphic at infinity* if and only if all the f_k as defined above are identically zero for $k < 0$, and is said to *go to* 0 *at infinity* if in addition f_0 is identically zero.

.4 Definition. Let $f : \Omega^r \to \mathbb{C}_\infty$ be a holomorphic Γ_U-invariant function.

We say that f is *bounded on vertical lines* if for every $\omega' \in \Omega^{r-1}$ there exist constants $\epsilon, R > 0$ such that if $\mathrm{d}(\omega_1, F_\infty^{r-1}\omega') > R$, then $|f(\omega)| < \epsilon$. If for every $\omega' \in \Omega^{r-1}$ and $\epsilon > 0$, there exists an $R > 0$ with this property, we say that f *tends to 0 on vertical lines.*

We say that f is *bounded* (resp. *tends to 0*) *on vertical cylinders* if for any $\omega' \in \Omega^{r-1}$ there exists an admissible neighbourhood $S \subset \Omega^{r-1}$ of ω' and $\epsilon, R > 0$ such that if $\mathrm{d}(\omega_1, F_\infty^{r-1}\omega') > R$ and $\omega' \in S$, then $|f(\omega)| < \epsilon$. (resp. if for all $\epsilon > 0$ there exists $R > 0$ with this property).

We now arrive at the result which will be used in Chapter 5:

.5 Theorem. *Let $f : \Omega^r \to \mathbb{C}_\infty$ be a holomorphic Γ-invariant function. The following conditions are equivalent:*

1. *f is bounded on vertical cylinders.*
2. *f is bounded on vertical lines.*
3. *f is holomorphic at infinity.*

Moreover, f goes to zero at infinity if and only if f tends to 0 *on vertical lines (or equivalently on vertical cylinders).*

The following proof is largely due to BBP.

Proof. **Item 1** $\implies$ **Item 2.** This is trivial.

Item 2 $\implies$ Item 3. By Proposition .2, there exists a sequence $(r_n > 0)_{n\geq 0}$ such that f is given by the u-expansion

6 $$f(\omega) = \sum_{k\in\mathbb{Z}} f_k(\omega')u_{\omega'}(\omega_1)^k,$$

which converge uniformly on the set

$$S_n := \left\{(\omega_1, \omega') \in \Omega^r \;\middle|\; (u_{\omega'}(\omega_1), \omega') \in \mathrm{B}(0, r_n) \times \Omega^{r-1}\right\}$$

for all sufficiently large $n \in \mathbb{N}_0$.

Let $\omega' \in \Omega^{r-1}$, $R > 0$, and $\epsilon > 0$ as in Definition .4. Suppose that $\omega_1 \in \mathbb{C}_\infty$ satisfies $\mathrm{d}(\omega_1, \omega' F_\infty^{r-1}) > R$; then we have $|u_{\omega'}(\omega_1)| < 1/R$.

Choosing $n \in \mathbb{N}$ sufficiently large and enlarging R if necessary, we may assume that $\omega_1, \omega') \in U_n$, that the expansion Equation 6 converges uniformly on U_n, and that $|f(\omega)| < \epsilon$ for all $\omega \in S_n$.

Now consider the Newton polygon of the series Equation 6, i.e. the boundary of the lower convex hull of the set of points $\left(k, -\log_q|f_k(\omega')|\right)$ in the Euclidean plane. [BBP1, Lemma 5.1] gives that $\lim\limits_{k\to-\infty} |f_k(\omega')|^{-1/k} = 0$, so that the slopes of the Newton polygon tend to $-\infty$ as $k \to -\infty$; thus either the series has a finite tail or infinitely many points lie on the Newton polygon for negative k.

Consider the line $y = mx + c$ with slope $m = \log_q|u_{\omega'}(\omega_1)|$ and tangent to the Newton polygon. By slightly perturbing ω_1, we may assume that this line touches the Newton polygon in only one point $(k, -\log_q|f_k(\omega')|)$. The corresponding term in Equation 6 then dominates the series, and the y-intercept of the line equals

$$c = -\log_q\left|f_k(\omega')u_{\omega'}(\omega_1)^k\right| = -\log_q|f(\omega)|.$$

Now if there exist points on the Newton polygon with $k < 0$, then by choosing $m = \log_q|u'_\omega|(\omega_1)$ sufficiently small (i.e. $d(\omega_1, \omega' F^{r-1})$ sufficiently large), we find that $|f(\omega)|$ can be made larger than the bound ϵ, i.e. f is not bounded on this vertical line, contradicting Item 2. Furthermore, if there exists a point with $k = 0$, then the same argument shows that $|f(\omega)| \geq |f_0(\omega')|$, so that f cannot vanish on the vertical line.

Item 3 $\implies$ Item 1. Let f have a u-expansion as before. Let $\omega' \in \Omega^{r-1}$ be given, and let $n \in \mathbb{N}$ such that $\omega' \in \Omega_n^{r-1} =: S$; here the Ω_n^{r-1} form an admissible covering of Ω^{r-1} by affinoid subsets. Let $r_n > 0$ be such that the u-expansion converges on

$$\{(\omega_1, \omega') \in \Omega^r \mid (u_{\omega'}(\omega_1), \omega') \in \mathrm{B}(0, r_n) \times \Omega_n^{r-1}\},$$

and set $R =: 1/r_n$.

Suppose that $(\omega_1, \omega')) \in \mathbb{C}_\infty \times S$ satisfies $\mathrm{d}(\omega_1, \omega' F_\infty^{r-1}) > R$; then $(u_{\omega'}(\omega_1), \omega') \in \mathrm{B}(0, r_n)' \times \Omega_n^{r-1}$ and so [BBP1, Lemma 5.1] gives

$$\liminf_{k \to \infty} |f_k(\omega')|^{1/k} < R.$$

Since also $|u_{\omega'}(\omega_1)| < 1/R$, we obtain

$$\liminf_{k \to \infty} \left| f_k(\omega') u_{\omega'}(\omega_1)^k \right| < 1,$$

and so $f(\omega)$ is bounded by some $N > 0$. Thus f is bounded on vertical cylinders.

Lastly, if f_0 is identically 0, then we may write

$$|f(\omega)| = |u_{\omega'}(\omega_1)| \cdot \left| \sum_{k \geq 0} f_{k+1}(\omega') u_{\omega'}(\omega_1)^k \right|,$$

where the sum on the right is bounded as before, and $|u_{\omega'}(\omega_1)| \to 0$ as $R \to \infty$, so f tends to 0 on vertical cylinders. $\square$